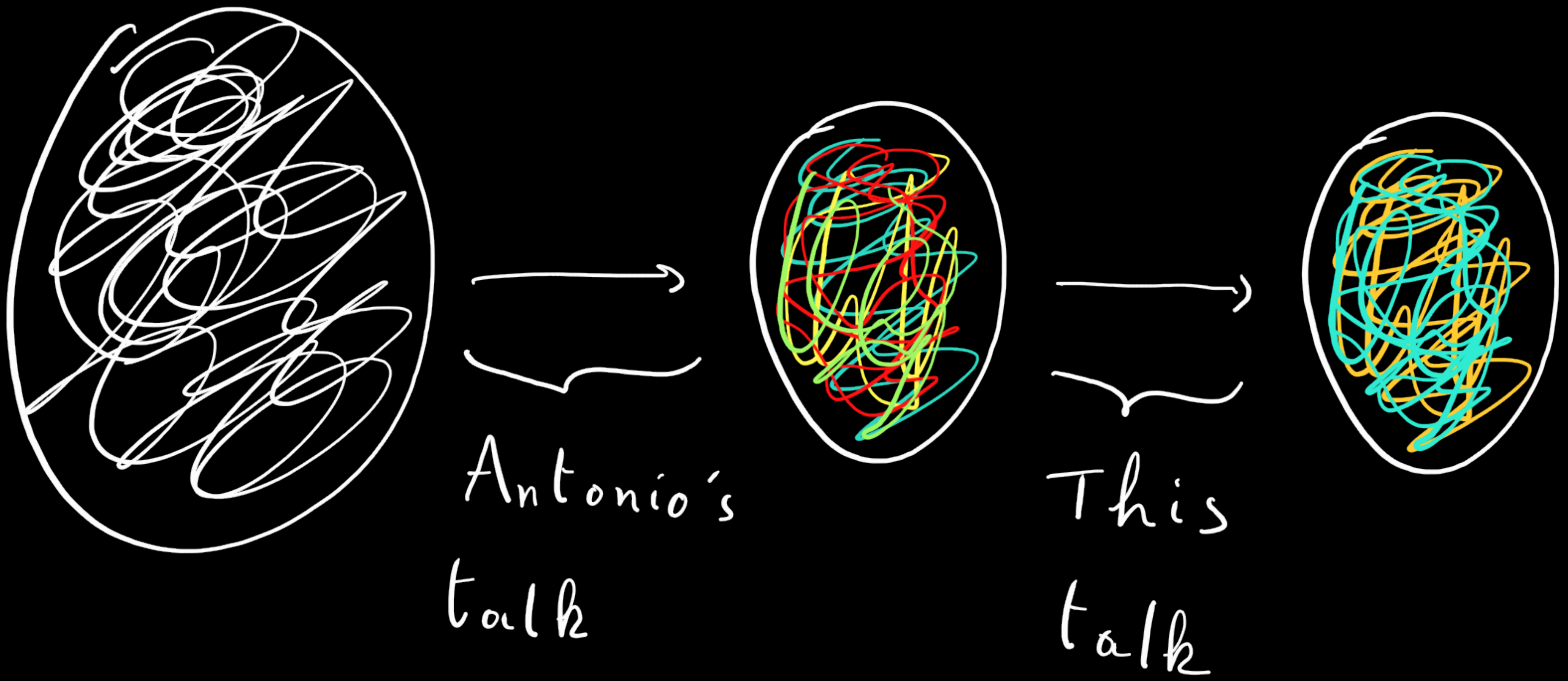
Simplifying acceptance conditions of ω-automata

Joint work with

Antonio Casares


Antonio's talk
This talk

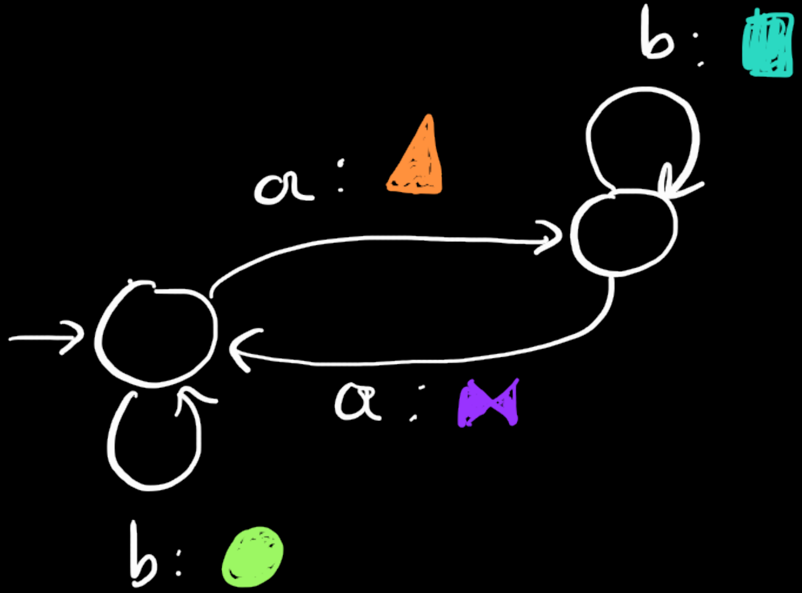
Muller automaton

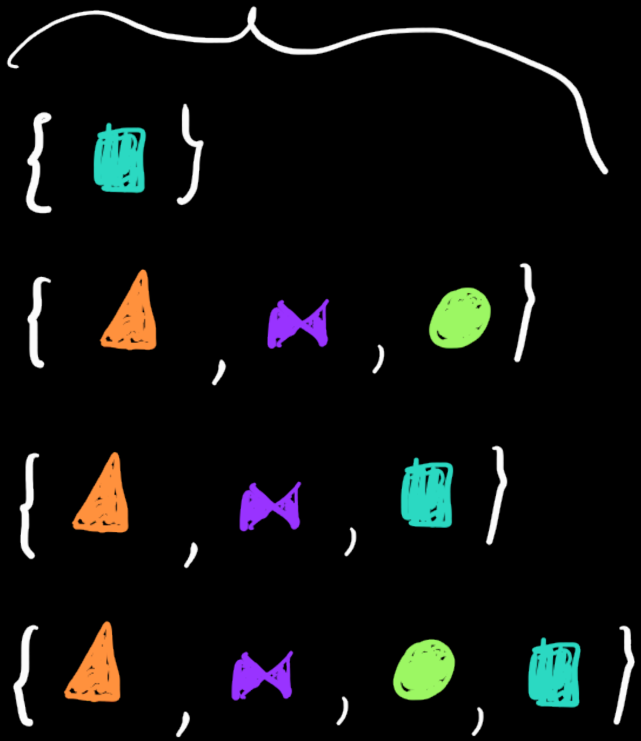
$$\mathcal{A} = (Q, \Sigma, \Delta, q_{init}, \underbrace{\Gamma}_{\text{Set of Colours}}, \underbrace{\lambda : \Delta \to \Gamma}_{\text{colouring function}}, \underbrace{Acc}_{\subseteq 2^{\Gamma}})$$

$w \in \mathcal{L}(\mathcal{A})$ if it has a run ρ s.t.

$\{c \in \Gamma \mid c$ appears ∞ many times in $\rho\} \in Acc$

Muller condition

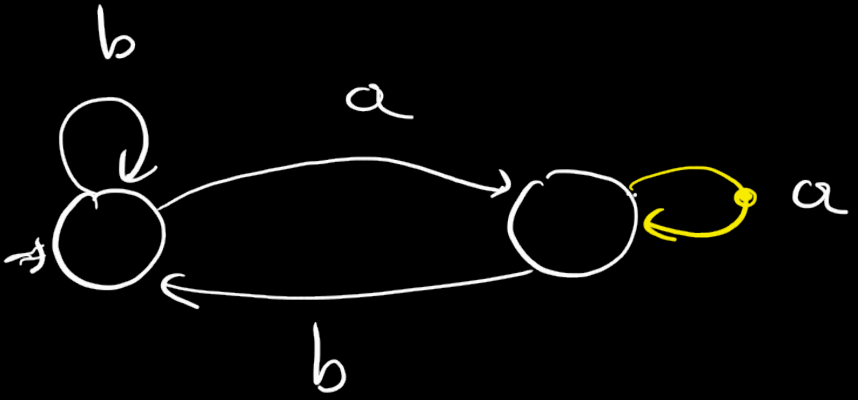




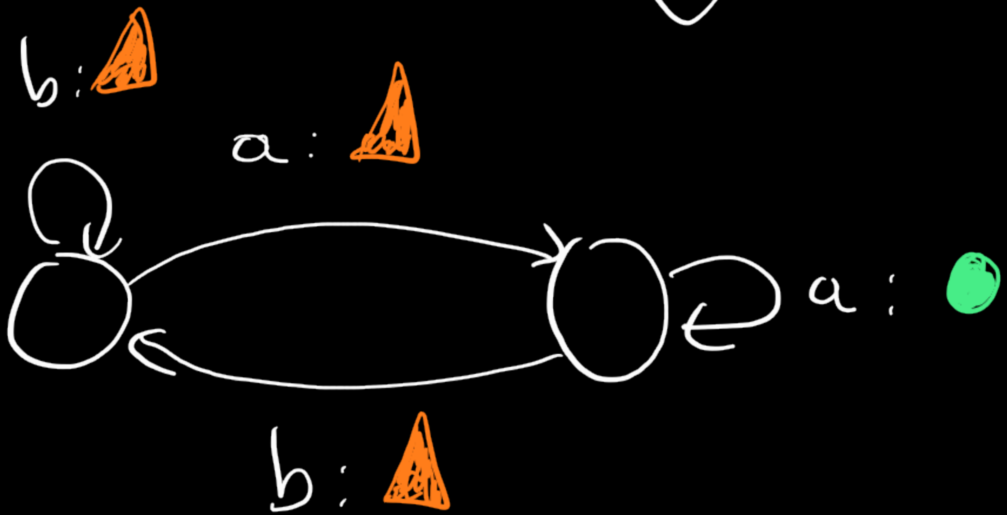
"Odd number of a or ∞ many a and b"

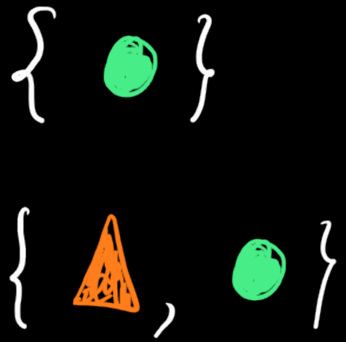
Büchi automaton for

"aa ∞ often"



⇓





Other types of conditions

Rabin: List of pairs of sets

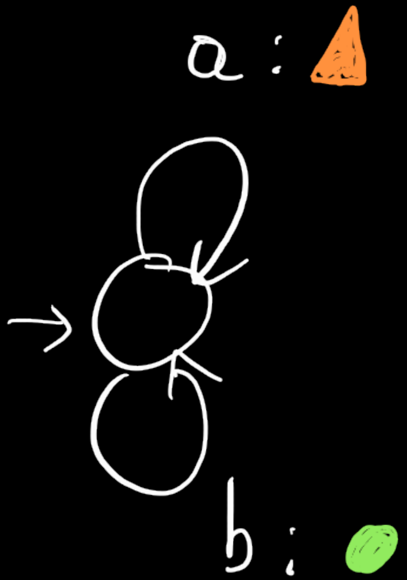
$$(G_1, R_1), \dots, (G_k, R_k)$$

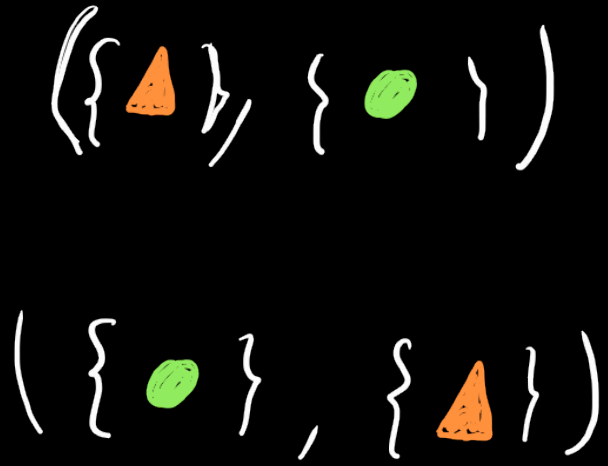
A set S is accepted if

$$\exists i,\ S \cap G_i \neq \emptyset \ \wedge\ S \cap R_i = \emptyset$$

⚠ Not all sets can be expressed this way

Example



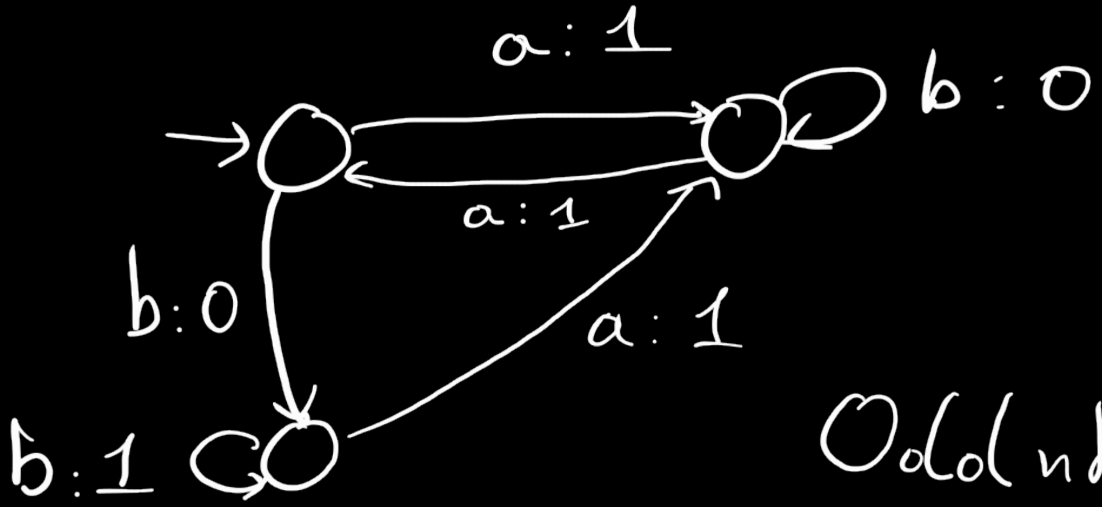


$$\{a, b\}^* a^\omega + b^\omega$$

Parity

Colours are $\{0, \dots, d\}$

S is accepted if $\min_{i \in S} i$ is even



Odd nb of a or ∞ many a and b

Problem 1 : Simplify conditions by themselves

Given two conditions C_1 and C_2 over sets of colours Γ_1 and Γ_2,

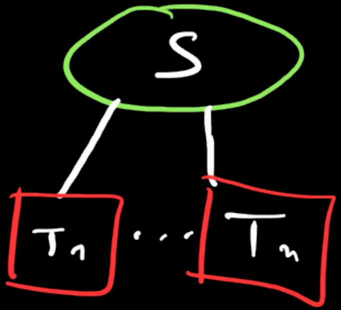
a translation from C_1 to C_2 is $\tau : \Gamma_1 \longrightarrow \Gamma_2$

$$\forall S \subseteq \Gamma_1, \quad S \models C_1 \iff \tau(S) \models C_2$$

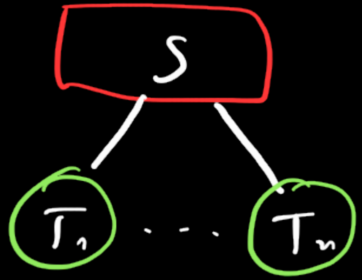
Given a Muller condition,

- Typing: Can it be translated to a Rabin/Parity condition?
- Colour minimisation: Can it be translated to a Muller condition with k colours?
- Rabin pair minimisation.

Zielonka tree of a Mullen condition

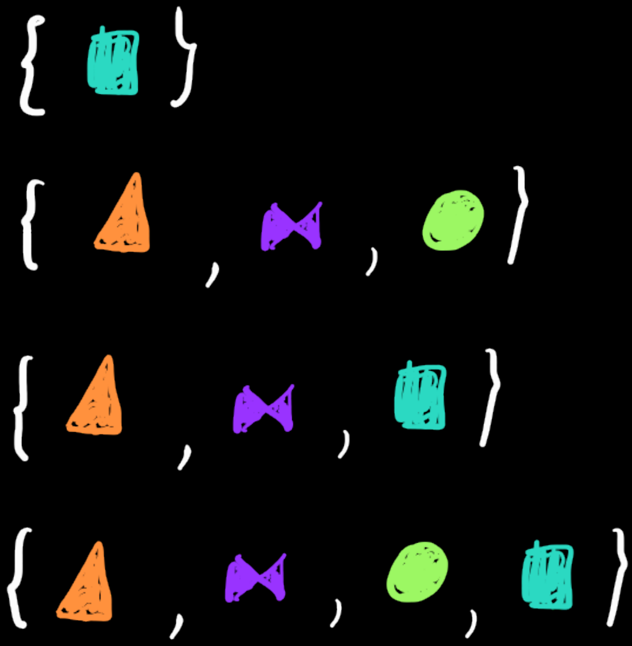
- Every node is or and labelled with a set of colours
- 

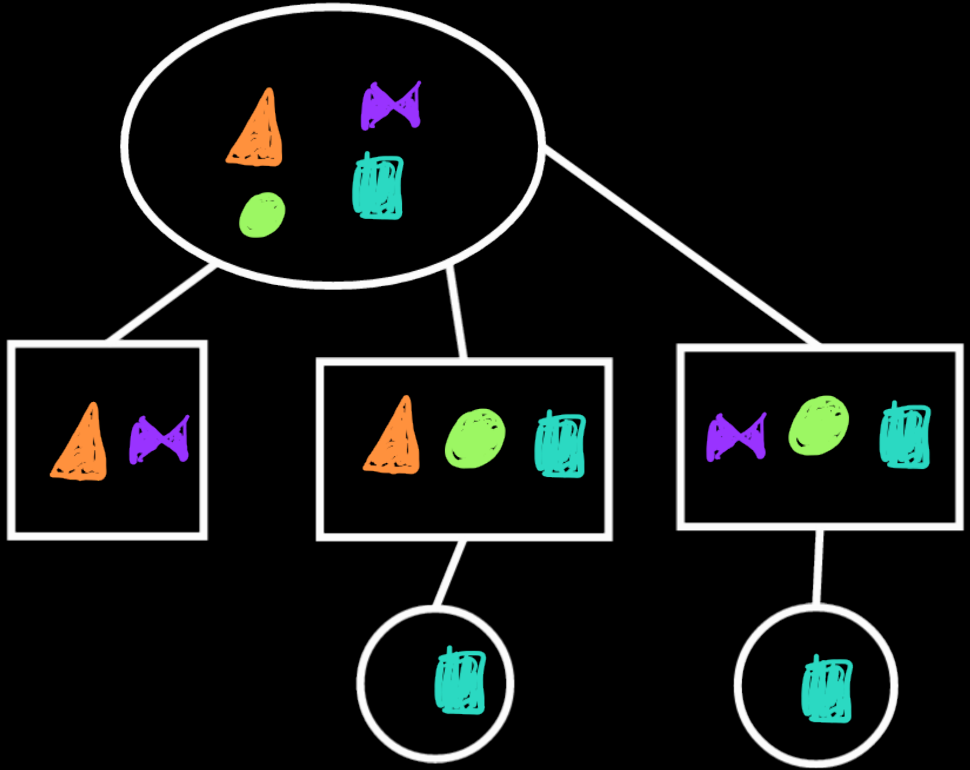
} max rejected subsets of S



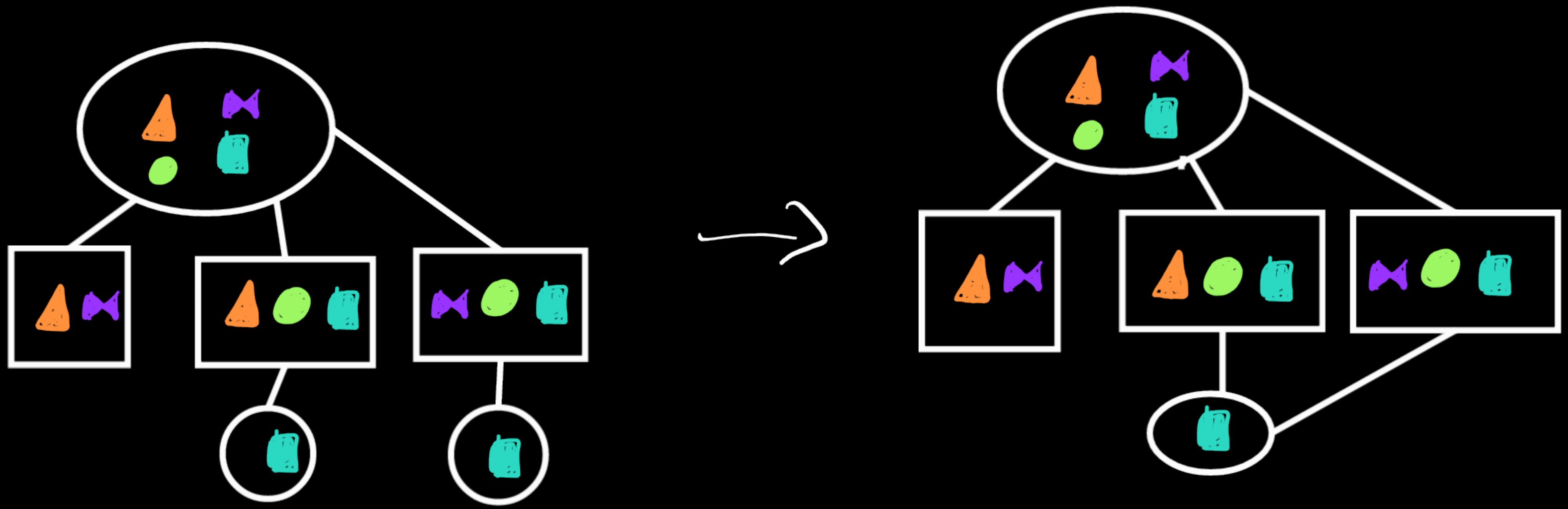
} max accepted subsets of S

Zielonka tree



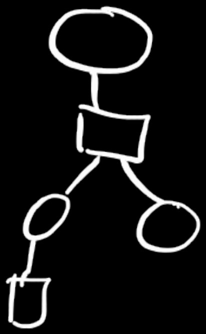


Zielonka DAG



Theorem [Zielonka '98]

- A Muller condition can be translated into a Rabin one iff every round node in its Z-tree* has ≤ 1 child



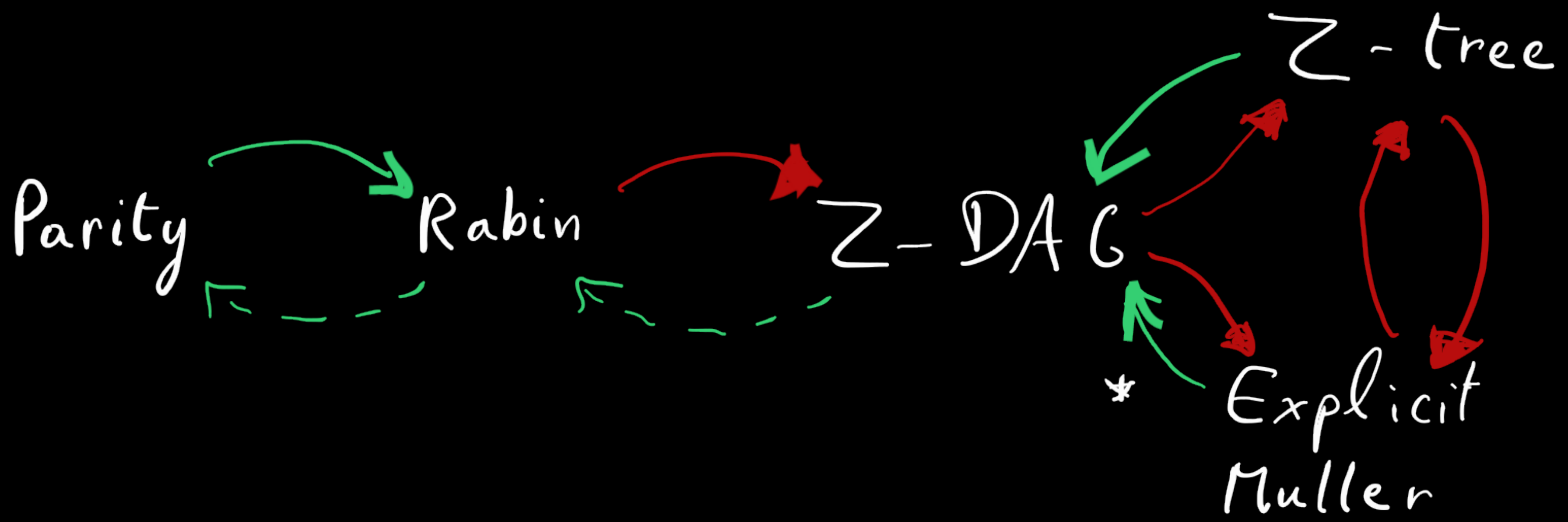
- It can be translated into a parity one iff every node has ≤ 1 child

* works with a Z-DAG

Cost of transformation

→ / - - → poly time

→ exp. time



* [Hunter, Dawar '05]

Theorem: Given a Muller condition we can minimise its number of colours in polynomial time.

Theorem: Given a Rabin condition we can minimise its number of pairs in polynomial time.

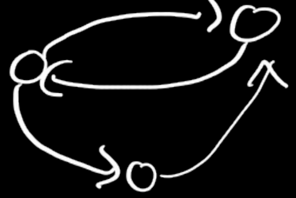
Problem 2: Simplify conditions on automata

Given $\mathcal{A}$ with colouring $\lambda : \Delta \to \Gamma$,
and condition C,
a relabelling is a function $\lambda' : \Delta \to \Gamma'$,
and C' such that

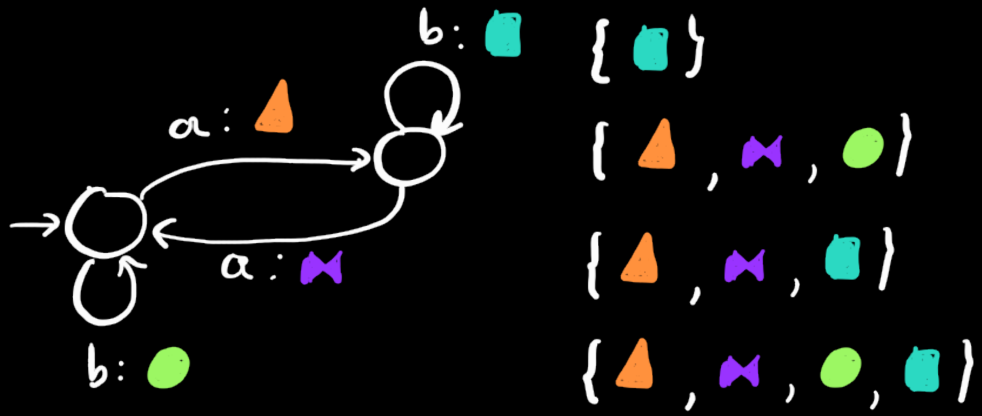
$$\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$$

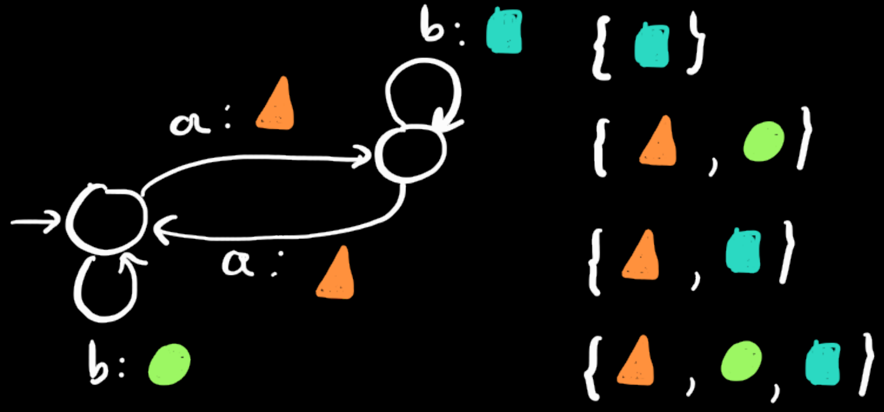
$\hookrightarrow$ $\mathcal{A}$ with λ' and C'

Cycle = strongly connected set of edges



<u>Remark:</u> (λ', c') is a relabelling of (λ, c) over $\mathcal{A}$ if they yield the same accepting cycles.


b:
a:
a:
b:

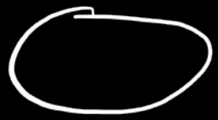

b:
a:
a:
b:

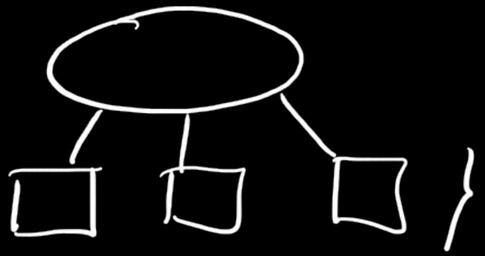
The Alternating Cycle Decomposition

~ The Zielonka tree of cycles

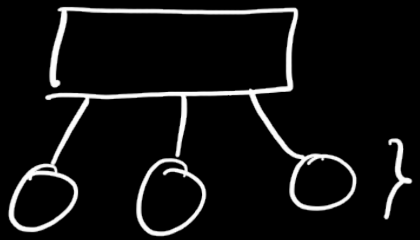
$\mathrm{ACD}(\mathcal{A})$ = forest where:

roots = maximal accepted cycles

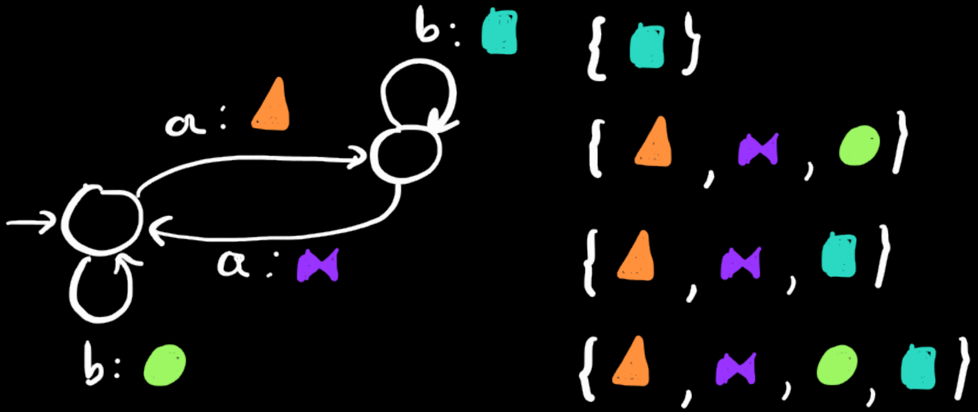


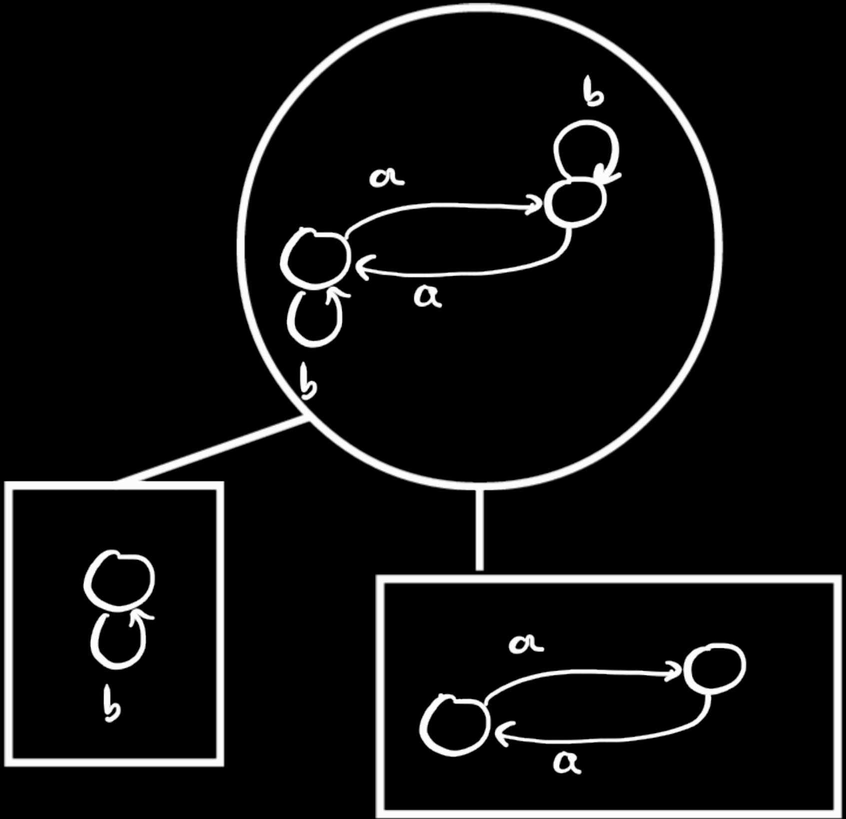


Max rejected subcycles



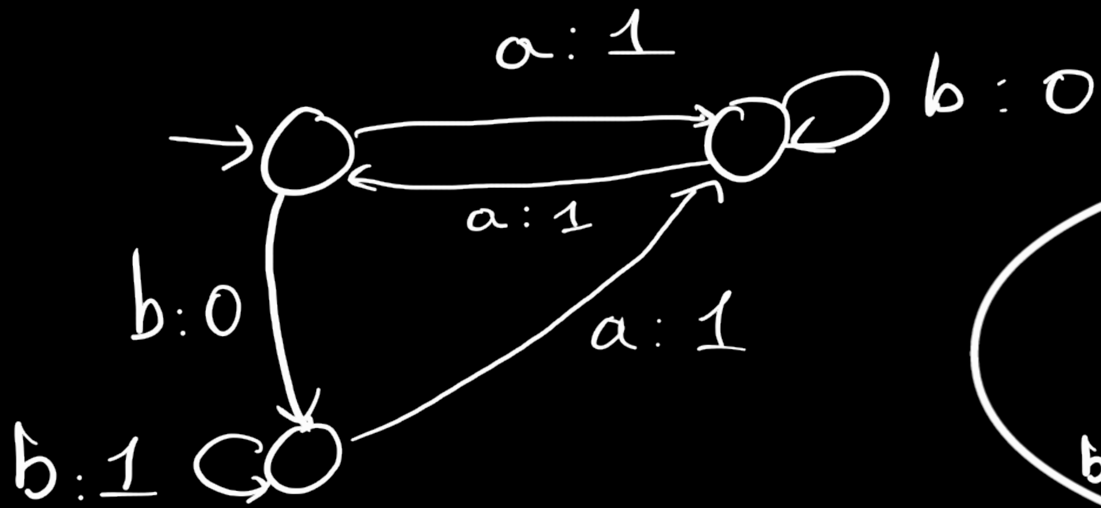
Max accepted subcycles.

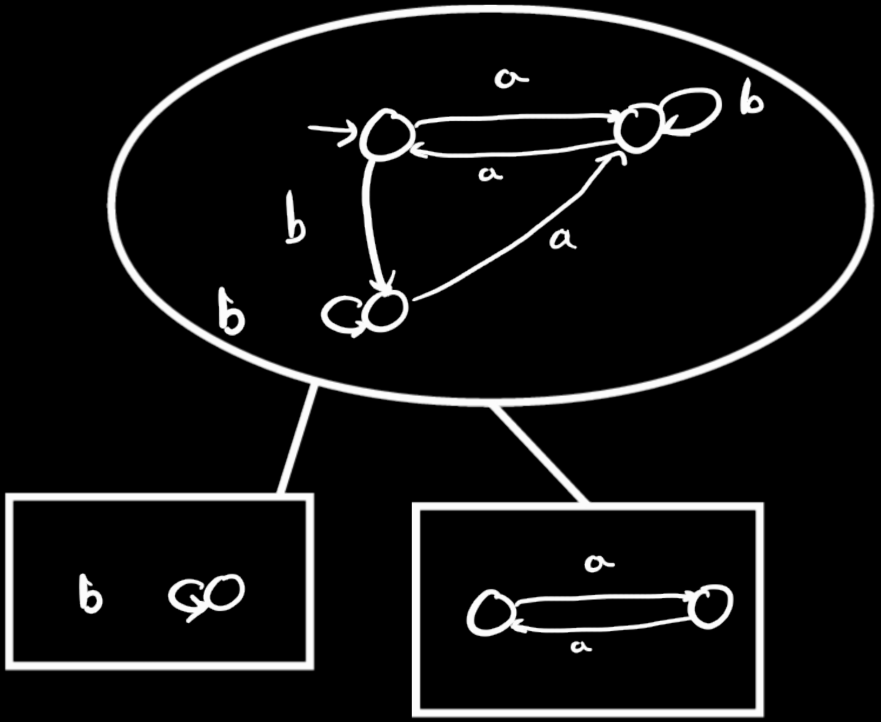

b :
a :
a :
b :


b
a
a
b
b
a
a

Typing [Casares, Colcombet, Fijalkow '21]

Given ACD($\mathcal{A}$), we can decide in PTIME if $\mathcal{A}$ can be relabelled with a parity or a Rabin condition.


a : 1
b : 0
a : 1
b : 0
a : 1
b : 1


a
b
a
b
a
b
b
a
a

Theorem: Given $\mathcal{A}$ a Muller automaton, we can compute ACD-DAG$(\mathcal{A})$ in polynomial time.

Corollary: We can decide in PTIME if we can relabel it with Rabin/Parity.

Good news: Number of colours can be minimised for Parity in polynomial time [Wilke, Yoo '96]

Bad news:

Minimising colours on Rabin pairs over an automaton is hard.

<u>Two reductions:</u>

First reduction

CLIQUE

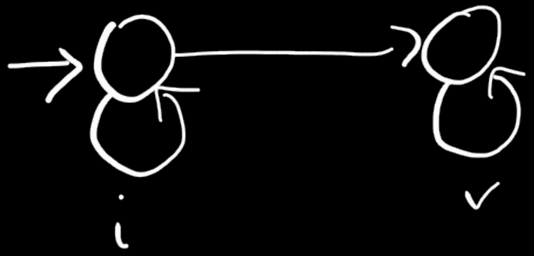
I : $G=(V,E)$, $k \in \mathbb{N}$

O : Does G have a k-clique?

$\longrightarrow$

I : $\mathcal{A}$ a Muller aut, $n \in \mathbb{N}$

O : Can we relabel $\mathcal{A}$ with n colours?



$1 \le i \le k$ $\qquad$ $v \in V$

$$\{ \{i,j\} \mid 1 \le i < j \le k \}$$
$$\cup \; \{ \{u,v\} \mid (u,v) \in E \}$$

We can relabel $\mathcal{A}$ with $|V|$ colours

$\iff$

G has a k-clique.

Second reduction

adapted from [Hugenroth '23]

GRAPH COLOURING

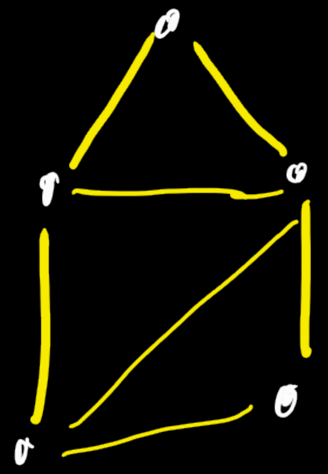
I: $G=(V,E)$, $k \in \mathbb{N}$

O: Is G k-colourable?

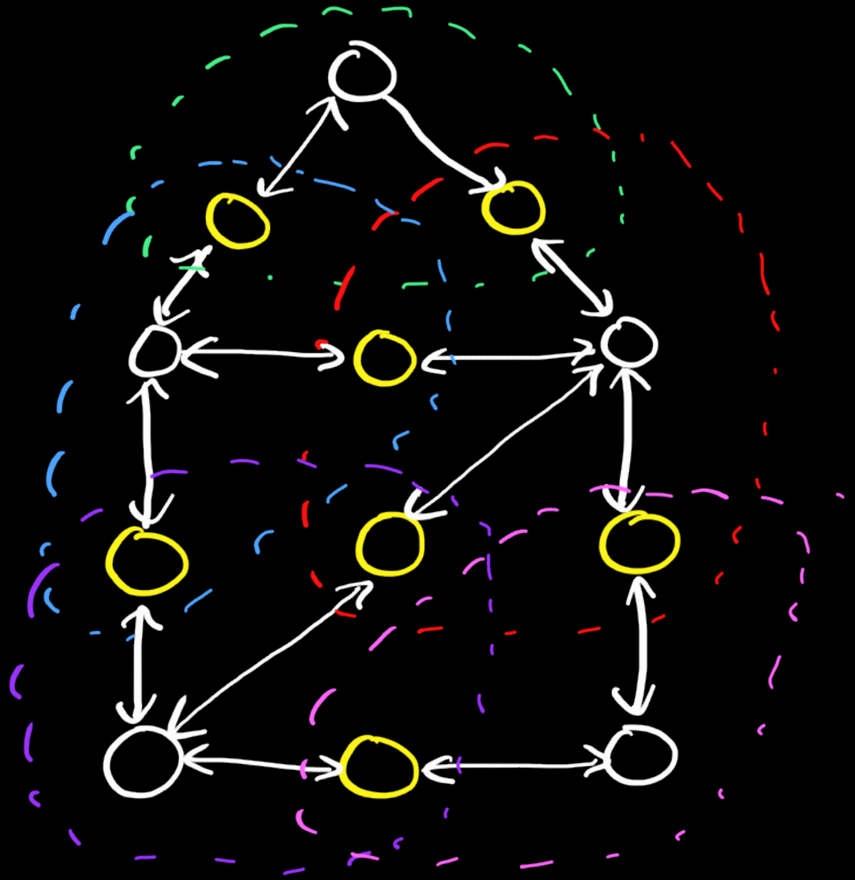
$\longrightarrow$

I: $\mathcal{A}$ a Rabin aut,
$n \in \mathbb{N}$

O: Can we relabel
$\mathcal{A}$ with n pairs?

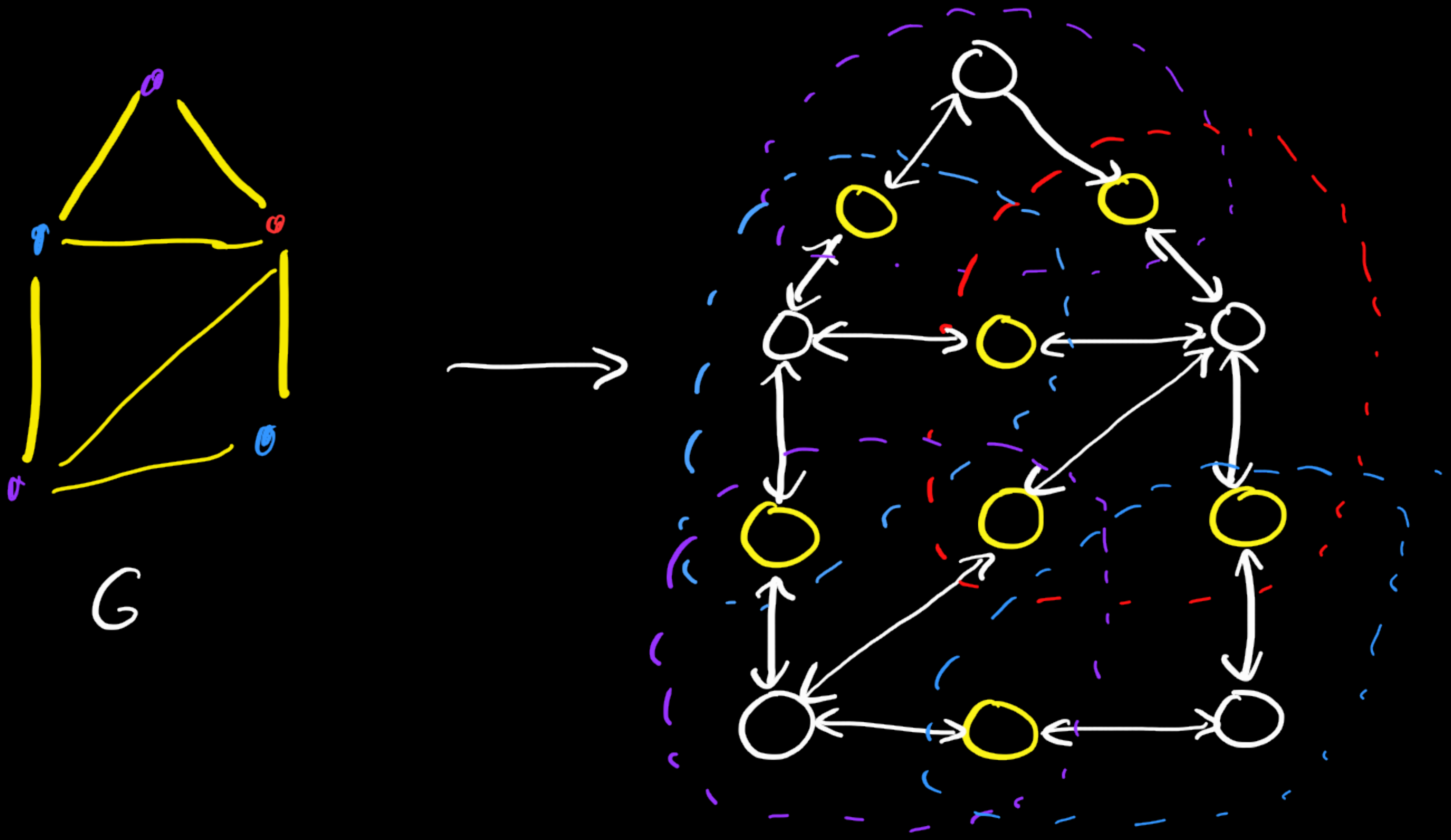


G



$\mathcal{A}$

Condition = stay within one of the regions



G $\mathcal{A}$

So . . .

- We can simplify and minimise conditions efficiently
- The ACD lets us decide the type of a condition on an automaton in PTIME.
- Minimising conditions on automata is NP-complete in general.