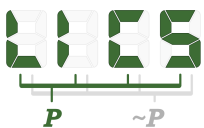
The complexity of downward closures of indexed languages

Richard Mandel **Corto Mascle** Georg Zetzsche

Max Planck Institute for Software Systems, Kaiserslautern



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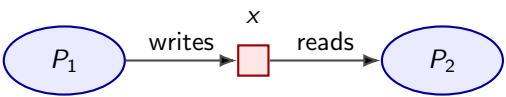

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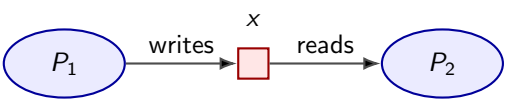

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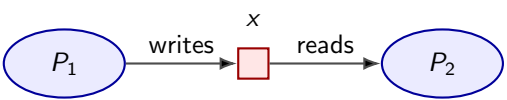

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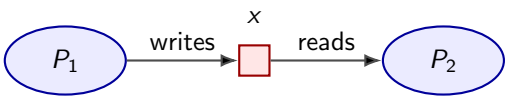

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Given a class of languages $\mathcal{C}$, is the downward-closure *computable* for these languages? How large is the resulting automaton?

Computing downward closures

Downward-closures are computable for:

- **Context-free languages** [van Leeuwen '78]
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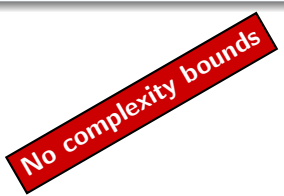

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No complexity bounds

Problem

Given a system $\mathcal{S}$ of size k in class $\mathcal{C}$, how large can an automaton recognising $L(\mathcal{S})\downarrow$ be?

Indexed grammars [Aho '68]

Indexed grammar= Context-free grammar where each non-terminal carries a stack.
N set of *non-terminals*, T set of *terminals*, Γ set of *stack symbols*.

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$$
\begin{aligned}
A &\to w \in T^* && \textit{terminal} \\
A &\to BC && \textit{copy} \\
A &\to B\gamma && \textit{push} \\
A\gamma &\to B && \textit{pop}
\end{aligned}
$$

Indexed grammars : Example

$$S \to S\gamma_\perp$$
$$S \to S\gamma_a$$
$$S \to S\gamma_b$$
$$S \to WW$$
$$W\gamma_a \to Wa$$
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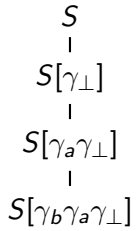

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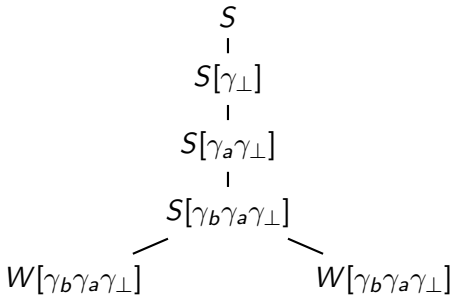

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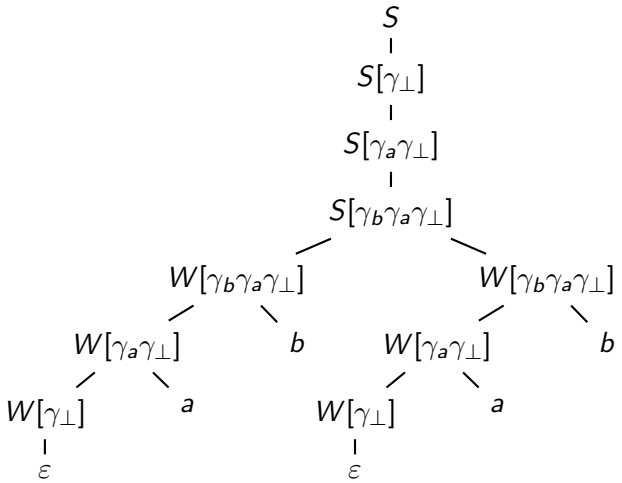

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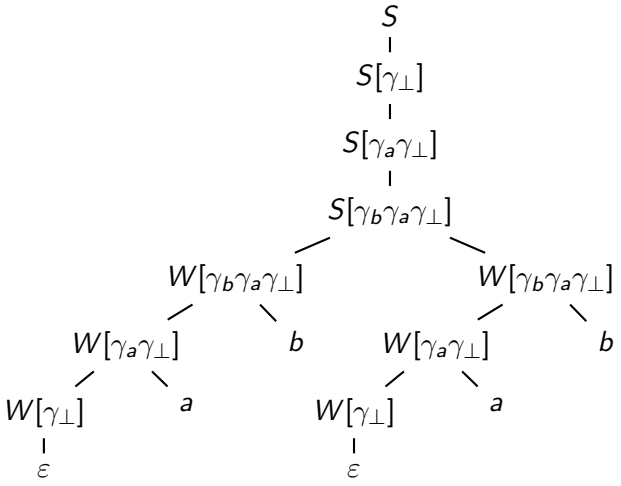
$$\{ww \mid w \in \{a, b\}^*\}$$

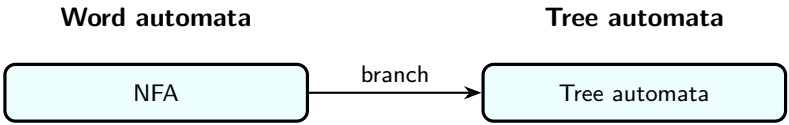


Higher-order pushdown automata

Word automata

Tree automata

Higher-order pushdown automata



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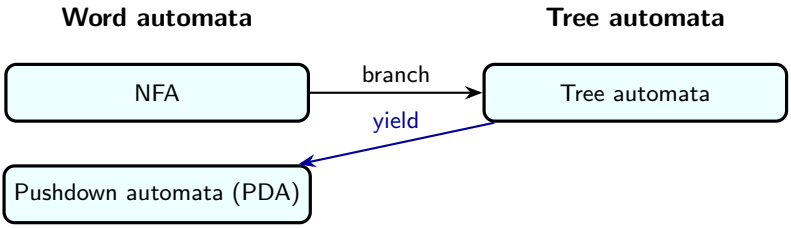

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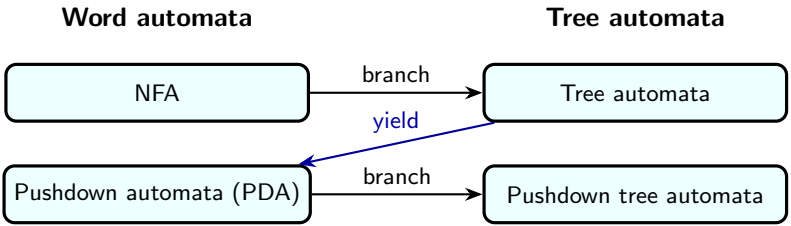

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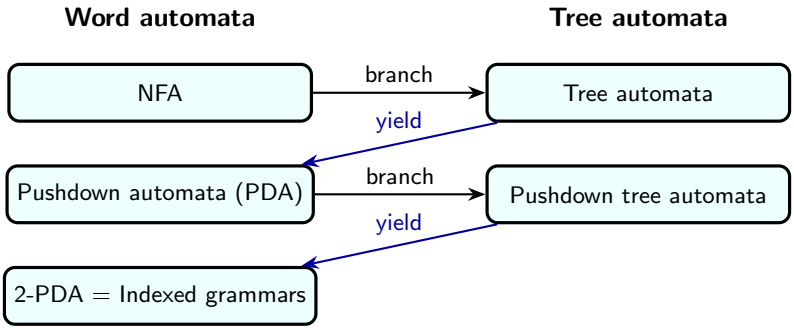

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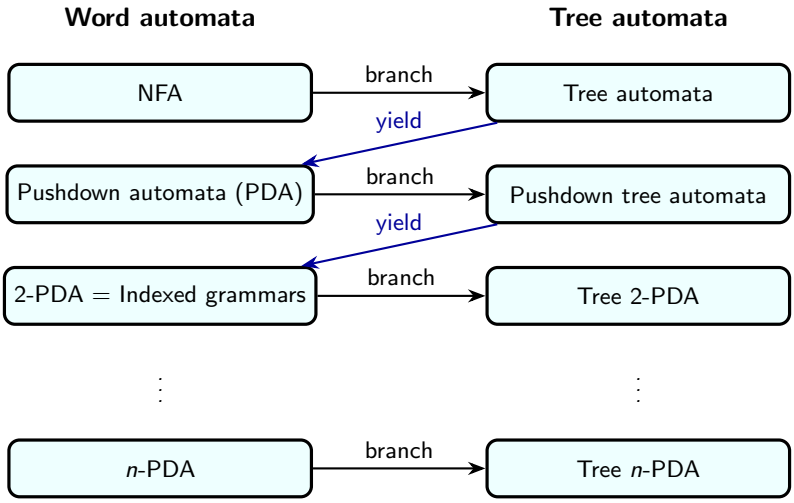

Our goal

Give tight bounds on the computation of downward-closures of indexed languages.

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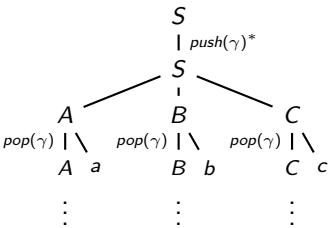
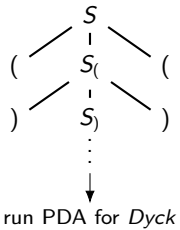
What can indexed grammars compute?

Examples

$\{a^n b^n c^n \mid n \in \mathbb{N}\}$



$\{uu^R \mid u \in Dyck_n\}$



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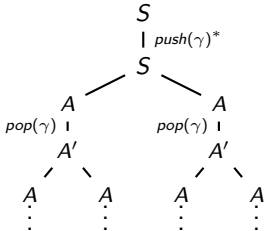

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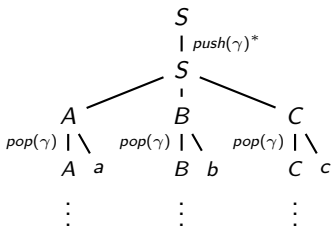

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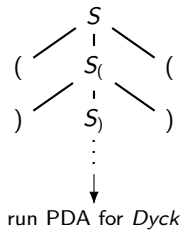

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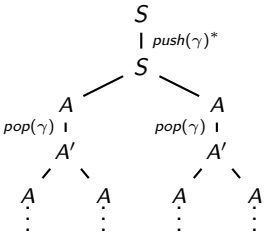

$L \subseteq \{a, \wedge, \vee, \neg, (,)\}^*$ set of satisfiable SAT formulas, with variable x_k encoded as a^k

Indexed grammars : Counting

[Aho '68]

If $\mathcal{G}$ accepts a word then it accepts a word of at most 3-exponential size.

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This work

There is $\mathcal{G}_k$ of size $\mathcal{O}(k)$ accepting the language $\{a^{2^{2^{2^k}}}\}$.

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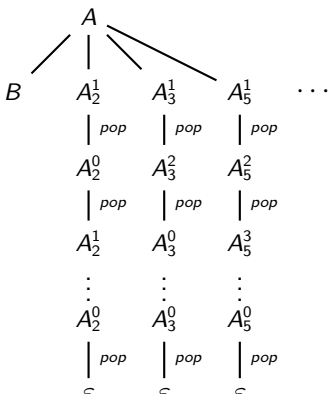
Corollary

There is an indexed grammar whose downward closure requires a 3-exponential NFA.

3-exponential construction

Check stack height



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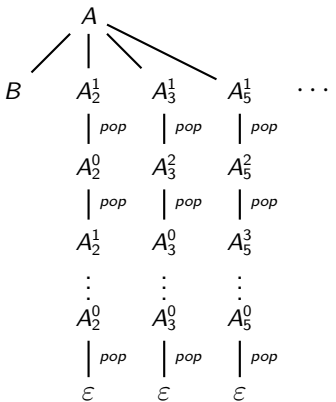

Binary counter

A
| $pop(1)^*$
B
| $pop(0)$
C
| $push(1)$
D
| $push(0)^*$
E
| check stack height 2^n
A
⋮

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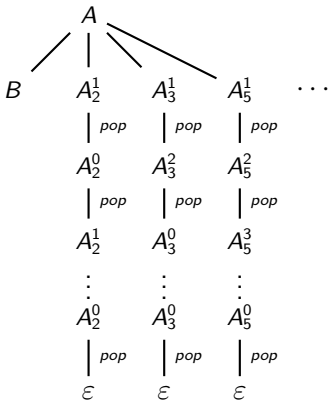

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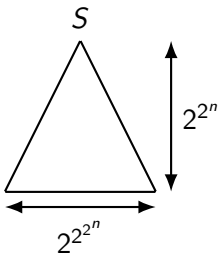
Branching



Upper bounds

Theorem [van Leeuwen '78, Courcelle '91]

The downward-closure of the language of a CFG is recognised by an NFA of exponential size.

Main result

The downward-closure of an indexed language is recognised by a CFG of 2-exponential size.

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Step 1: Add new rules

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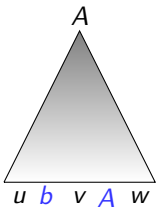

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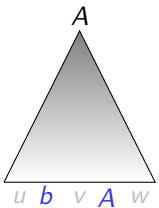

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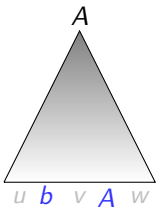

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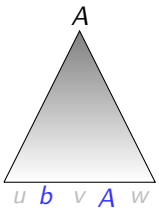

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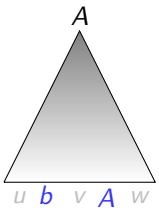

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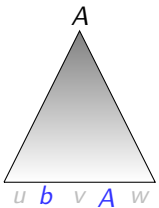

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Step 2: Simulation with an NFA

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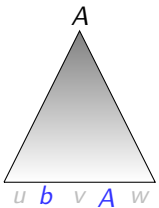
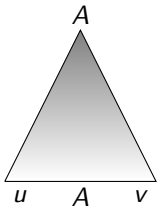


Step 2: Simulation with an NFA



$\Gamma_l(A) = \{b \mid A \text{ derives } w \in \Sigma^* b \Sigma^* A \Sigma^*\}$.
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New rule: $A \to xAy$ for all $x \in \Gamma_l(A)^*, y \in \Gamma_r(A)^*$
Preserves the downward-closure.

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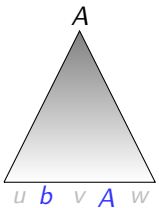

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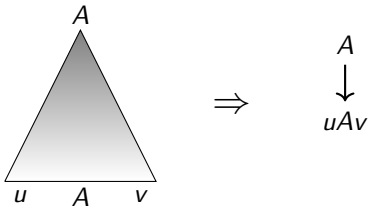

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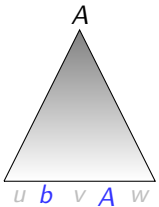

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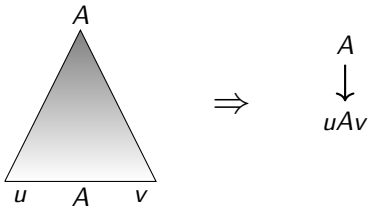

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Those can be simulated by a finite-state automaton.

Upper bounds

Theorem [van Leeuwen '78, Courcelle '91]

The downward-closure of the language of a CFG is recognised by an NFA of exponential size.

Main result

The downward-closure of an indexed language is recognised by a CFG of 2-exponential size.

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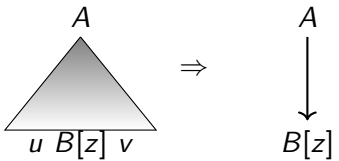
If A can produce $uB[z]v$ then $A \to B[z]$ is a sound rule.



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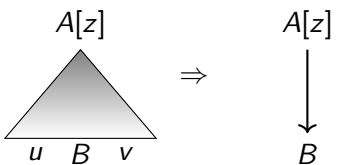

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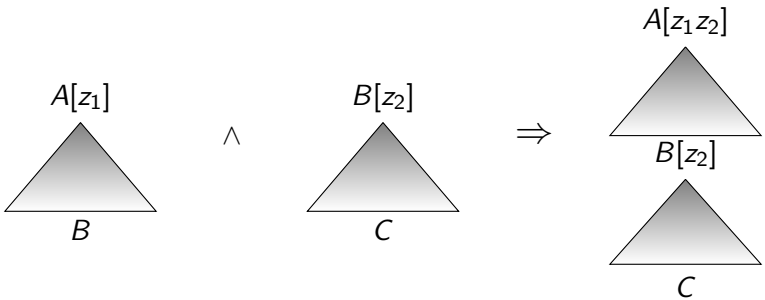
M_{push} and M_{pop} are **morphisms**: $M_{pop}(z_1)M_{pop}(z_2) = M_{pop}(z_1 z_2)$.



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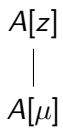

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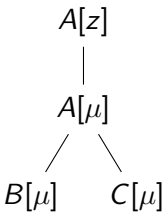

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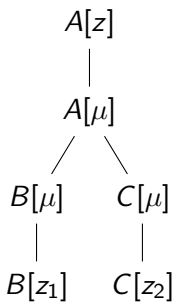

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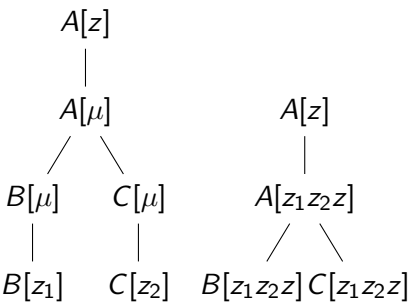

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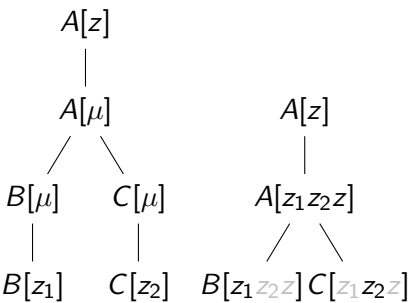

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We can define *summaries* representing "folded" words so that:

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\end{aligned}
\qquad \Rightarrow \qquad
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Theorem

This CFG has the same downward closure as the indexed grammar.

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Composed with the construction for context-free grammars:

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An open problem

Parikh image

$$w = acbbabcb$$
$$\Downarrow$$
$$\psi(w) = 2a + 4b + 2c$$

Parikh's theorem

For all context-free grammar $\mathcal{G}$ there is a regular language with the same Parikh image.

What about indexed languages?

Thank you for your attention!

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