Keyboards as a new model of computation

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September 23, 2021

Context

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We try to write the word `bip`.

1. We press the `b` key: it writes “bis”.
2. We press the `i` key: it erases two letters, writes “lo” and moves the cursor to the left.
3. We press the `p` key: it moves the cursor to the right and writes “op”.

“bloop|”

Instead of “bip”, the keyboard wrote “bloop”!

What do we do?

We could try to fix the keyboard...

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...or we could try to see what we can do with it! Can we write any word? If not, which words can we write?

Modelling

Atomic operations

- `a` for $a \in \Sigma$: writes "a" to the left of the cursor.

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- A keyboard is a finite set of keys.

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Keyboard

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Our broken keyboard

We wrote “bloop” by pressing three keys:

$$\{\mathtt{bis}, \leftarrow\leftarrow\mathtt{lo}\blacktriangleleft, \blacktriangleright\mathtt{op}\}.$$

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$$\langle u|v\rangle \cdot a = \langle ua|v\rangle \text{ if } a \text{ is a letter.}$$

$$\langle \varepsilon|v\rangle \cdot \leftarrow = \langle \varepsilon|v\rangle \quad \text{and} \quad \left\langle u'a\middle|v\right\rangle \cdot \leftarrow = \left\langle u'\middle|v\right\rangle$$

$$\langle \varepsilon|v\rangle \cdot \blacktriangleleft = \langle \varepsilon|v\rangle \quad \text{and} \quad \left\langle u'a\middle|v\right\rangle \cdot \blacktriangleleft = \left\langle u'\middle|av\right\rangle$$

$$\langle u|\varepsilon\rangle \cdot \blacktriangleright = \langle u|\varepsilon\rangle \quad \text{and} \quad \left\langle u\middle|av'\right\rangle \cdot \blacktriangleright = \left\langle ua\middle|v'\right\rangle$$

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We apply $t = \leftarrow a \blacktriangleright$ to $\langle c|d \rangle$.

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$$\begin{aligned}\langle c|d\rangle &\xrightarrow{\leftarrow} \langle \varepsilon|d\rangle \\ &\xrightarrow{a} \langle a|d\rangle \\ &\xrightarrow{\blacktriangleright} \langle ad|\varepsilon\rangle.\end{aligned}$$

Hence $\langle c|d\rangle \xrightarrow{t} \langle ad|\varepsilon\rangle$.

Language

The language of a keyboard K is the set of words we can obtain from configuration $\langle \varepsilon | \varepsilon \rangle$ by applying a sequence of keys from K,

$$\mathcal{L}(K) = \left\{ uv \;\middle|\; \exists t_1, \ldots, t_n \in K, \langle \varepsilon | \varepsilon \rangle \xrightarrow{t_1 \ldots t_n} \langle u | v \rangle \right\}.$$

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Let $t_1 = \texttt{bis}, t_2 = \leftarrow\leftarrow\texttt{lo}\blacktriangleleft, t_3 = \blacktriangleright\texttt{op}$ and $K = \{t_1, t_2, t_3\}$.

$$\begin{aligned} \langle \varepsilon | \varepsilon \rangle &\xrightarrow{t_1} \langle bis | \varepsilon \rangle \\ &\xrightarrow{t_2} \langle bl | o \rangle \\ &\xrightarrow{t_3} \langle bloop | \varepsilon \rangle \end{aligned}$$

The word “bloop” is in the language of K.

Some examples

- The language of $K = \{\texttt{ab}, \texttt{a}\}$?

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  The Dyck language (correctly nested sequences of brackets)!

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- A keyboard for ab^+?

$$K = \{\leftarrow ab \blacktriangleleft\}$$

Keyboards expressivity

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- $\left\{a^{2n+1} \;\middle|\; n \in \mathbb{N}\right\}$?

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Add an "Entry"!

An "Entry" symbol ■ which validates the word!

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■ is useful!

The language $\left\{ a^{2n+1} \;\middle|\; n \in \mathbb{N} \right\}$ is recognized by $\{aa, a\blacksquare\}$.

Two types of keyboards

- Keyboards with entry are called **manual**.
- Keyboards without entry are called **automatic**.

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Theorem (Simulation)

The language of an automatic keyboard K_A is also recognized by the (manual) keyboard

$$K_M = \{t \mid t \in K_A\} \cup \{t\blacksquare \mid t \in K_A\}.$$

Edge effects

The action of a key may differ when the cursor is close to an end of the word!

An automatic keyboard for $\left\{a^{2n+1} \;\middle|\; n \in \mathbb{N}\right\}$

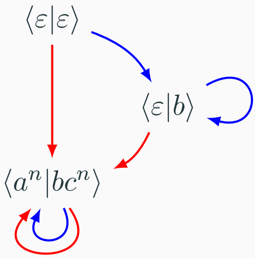
This language is recognized by the keyboard $\{t_1 = \leftarrow \texttt{a}, t_2 = \leftarrow \texttt{aaa}\}$.

$$\langle \varepsilon | \varepsilon \rangle \xrightarrow{t_1} \langle a | \varepsilon \rangle \qquad \left\langle a^{2n+1} \middle| \varepsilon \right\rangle \xrightarrow{t_1} \left\langle a^{2n+1} \middle| \varepsilon \right\rangle$$

$$\langle \varepsilon | \varepsilon \rangle \xrightarrow{t_2} \langle aaa | \varepsilon \rangle \qquad \left\langle a^{2n+1} \middle| \varepsilon \right\rangle \xrightarrow{t_2} \left\langle a^{2n+3} \middle| \varepsilon \right\rangle$$

The language $a^n bc^n$

$L = a^n bc^n$ is recognized by $K = \{$b▶◀←, ▶←abc◀◀$\}$.



Starting from $\langle \varepsilon | \varepsilon \rangle$, b▶◀← writes a b, and from $\langle a^n | bc^n \rangle$ it doesn't do anything.

Invariant: we are always in a configuration of the form $\langle a^n | bc^n \rangle$.

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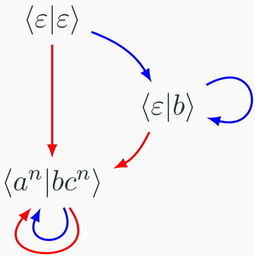

$$
\begin{aligned}
\langle \varepsilon | \varepsilon \rangle &\xrightarrow{t_2} \langle a | bc \rangle \\
&\xrightarrow{t_2} \langle aa | bcc \rangle \\
&\xrightarrow{t_1} \langle aa | bcc \rangle \\
&\xrightarrow{t_2} \langle aaa | bccc \rangle
\end{aligned}
$$

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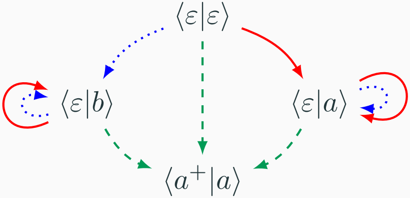
The language $a^* + b$

$L = a^* + b$ is recognized by $K = \{$b▶◀←, a▶◀←, ▶←aa◀$\}$.



From a configuration of the form $\langle a^n | a \rangle$, b▶◀← and a▶◀← have no effect, but ▶←aa◀ adds an a and leads to $\langle a^{n+1} | a \rangle$.

Classes of languages

- B: with $\leftarrow$
- E: with $\blacksquare$
- L: with $\blacktriangleleft$
- A: with $\blacktriangleright$ and $\blacktriangleleft$

$\mathsf{MK} : \{\}$	$\mathsf{LK} : \{\blacktriangleleft\}$	$\mathsf{AK} : \{\blacktriangleleft, \blacktriangleright\}$
$\mathsf{EK} : \{\blacksquare\}$	$\mathsf{LEK} : \{\blacktriangleleft, \blacksquare\}$	$\mathsf{AEK} : \{\blacktriangleleft, \blacktriangleright, \blacksquare\}$
$\mathsf{BK} : \{\leftarrow\}$	$\mathsf{BLK} : \{\blacktriangleleft, \leftarrow\}$	$\mathsf{BAK} : \{\blacktriangleleft, \blacktriangleright, \leftarrow\}$
$\mathsf{BEK} : \{\leftarrow, \blacksquare\}$	$\mathsf{BLEK} : \{\blacktriangleleft, \leftarrow, \blacksquare\}$	$\mathsf{BAEK} : \{\blacktriangleleft, \blacktriangleright, \leftarrow, \blacksquare\}$

Visiting the zoo

BEK{$\leftarrow$, ■}: A gentle animal

Lemma

For all $c \in A$, $c\leftarrow$ *is equivalent to* ε.

BEK{←, ■}: A gentle animal

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Simplification

$$
\begin{aligned}
\leftarrow abb\leftarrow ba\leftarrow^3 &\iff \leftarrow abb\leftarrow ba\leftarrow^3 \\
&\iff \leftarrow abba\leftarrow\leftarrow^2 \\
&\iff \leftarrow abb\leftarrow\leftarrow \\
&\iff \leftarrow ab\leftarrow \\
&\iff \leftarrow a
\end{aligned}
$$

BEK$\{\leftarrow, \blacksquare\}$: A gentle animal

Lemma (BEK normal form)

Every key of BEK is equivalent to a key of the form $\leftarrow^* A^*$.

Further, as we start on the empty configuration and never apply any ◀, the cursor is always on the right end of the word.

Lemma

Applying a sequence of keys of BEK from a configuration $\langle w|\varepsilon\rangle$ *yields a configuration of the form* $\langle w'|\varepsilon\rangle$.

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$$\leftarrow^m a_1 \cdots a_n b_1 \cdots b_p$$

BEK$\{\leftarrow, \blacksquare\}$: **Proof of rationality**

Say we want to write abc with the keyboard
$K = \{\leftarrow^2 aba, \leftarrow^4 bc\}$:

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→ We only care about the **number** of letters that will be erased, not about the word they form!

BEK$\{\leftarrow, \blacksquare\}$: **Proof of rationality**

To each key we can associate its numerical *trace*, which is the number of letters it writes minus its number of $\leftarrow$.

$$\leftarrow^2 aba \longmapsto +1$$
$$\leftarrow^4 bc \longmapsto -2$$

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Let p be the gcd of all the traces of keys of K.

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Proposition

We can turn i extra letters into j extra letters if and only if p divides $|i-j|$ (up to some minor conditions).

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We construct an NFA with states $\{0, \ldots, n\}$, n being the maximal length of a key of K.

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- $i \xrightarrow{u} j$ simulates the application of a key of the form $\leftarrow^{i} uv$ with $|v| = j$.

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- $i \xrightarrow{u} j$ simulates the application of a key of the form $\leftarrow^{i} u v$ with $|v| = j$.
- $i \xrightarrow{\varepsilon} j$ simulates the application of a series of keys not affecting the permanent letters but switching the number of extra letters from i to j.

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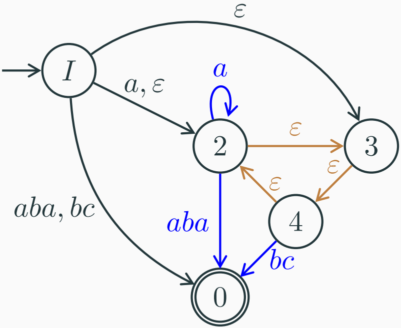

$$L(K) = a^*(aba + bc)$$

BLEK$\{\blacktriangleleft, \leftarrow, \blacksquare\}$: A ferocious creature?

The problem with BLEK

The left arrow allows for modifications anywhere in the word!

For instance, $\blacktriangleleft^3\leftarrow$ allows one to erase letters inside the word.

BLEK{◀, ←, ■}: A tamed creature

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For instance, $◀^3←$ allows one to erase letters inside the word.

Not so fast!

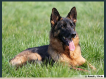
The letters to the right of the word are “fixed”.

$$\langle u | v \rangle \xrightarrow{a} \langle ua | v \rangle$$

$$\langle ua | v \rangle \xrightarrow{◀} \langle u | av \rangle$$

$$\langle ua | v \rangle \xrightarrow{\leftarrow} \langle u | v \rangle$$

BLEK$\{\blacktriangleleft, \leftarrow, \blacksquare\}$: A tamed creature



Lemma (A property of BLEK)

Any sequence of keys of BLEK *applied from a configuration* $\langle u|v\rangle$ *leads to a configuration of the following form:* $\langle u'|wv\rangle$.

The left arrow can be interpreted as a way to record the letter to the left of the cursor.

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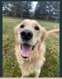

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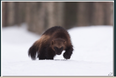
Theorem

For all keyboard K *of* BLEK *there exists a pushdown automaton of polynomial size recognising* $\mathcal{L}(K)$.

AEK{◄, ►, ■}: A wild being

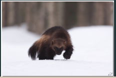


No more erasing, we only add letters!

Lemma (Monotony)

Applying any sequence of keys of AEK *to a configuration* $\langle u|v\rangle$ *yields a configuration* $\langle u'|v'\rangle$ *with* $|u'| + |v'| \geq |u| + |v|$.

AEK$\{\blacktriangleleft, \blacktriangleright, \blacksquare\}$: A wild being



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Theorem

For all keyboard K *of* AEK *there exists a linear bounded automaton of polynomial size recognising* $\mathcal{L}(K)$.

BAEK$\{\blacktriangleleft, \blacktriangleright, \leftarrow, \blacksquare\}$: The monster

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Proposition

Since a key can only modify the size of a configuration in a bounded way, if w is accepted, then some slightly smaller or longer word is also accepted.

Application

$\left\{ \mathsf{a}^{n^2} \,\middle|\, n \in \mathbb{N} \right\}$ and $\{\mathsf{a}^p \mid p \text{ prime}\}$ are not recognised by any keyboard.

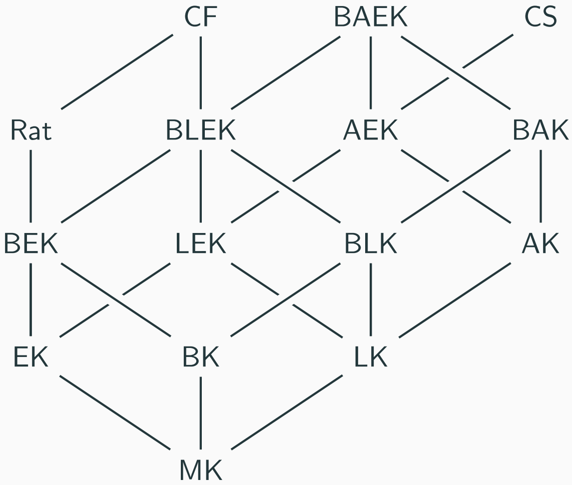
The keyboard hierarchy

Strict hierarchy theorem

Theorem

- *All 12 keyboard language classes we considered are distinct.*
  In particular, not all keyboards are automatic!
- *The only inclusions between classes are trivial ones*
  (except possibly for the inclusions of EK *and* BEK *in* BAK*).*

A strict hierarchy



	Membership	Universality
MK	P	P
EK	P	P
BK	P	coNP
BEK	P	PSPACE
LK	P	?
LEK	P	?
BLK	P	?
BLEK	P	?
AK	NP	?
AEK	NP	?
BAK	?	?
BAEK	?	?

	Complement	Concatenation	Intersection
MK	a^{2n}	a^*c^*	$(ab+bb+ba)^* \cap (ba+b)^*$
EK	a^{2n+3}	a^*c^*	$(ab+bb+ba)^* \cap (ba+b)^*$
BK	$(a+b)^*$ with $\|A\| = 3$	a^*c^*	$\mathcal{L}(K_1) \cap \mathcal{L}(K_2)$
BEK	$(a+b)^*$ with $\|A\| = 3$	a^*c^*	$\mathcal{L}(K_1) \cap \mathcal{L}(K_2)$
LK	a^{2n}	$a^n b^n c^m d^m$	$a^n b^n c^n$
LEK	a^{2n+3}	$a^n c a^n a^m c a^m$	$a^n b^n c^n$
BLK	$\{w \mid \|w\|_a \leq 1\}$	$(aa)^*(b+b^2)$	$a^n b^n c^n$
BLEK	$\{w \mid \|w\|_a \leq 1\}$	$a^n c a^n a^m c a^m$	$a^n b^n c^n$
AK	a^{2n}	$a^n b^n c^m d^m$	$a^n b^n c^n$
AEK	a^{2n+3}	$a^n c a^n a^m c a^m$	$a^n b^n c^n$
BAK	?	$a^n c a^n a^m c a^m$	$a^n b^n c^n$
BAEK	?	$a^n c a^n a^m c a^m$	$a^n b^n c^n$

	Mirror	Morphism	Union
MK	✓	✓	$a^* + b^*$
EK	b^*a	✓	$a^* + b^*$
BK	b^*a	$(a^2)^*(b+c)$	$a^* + b^*$
BEK	b^*a	?	$a^* + b^*$
LK	$b^n c(ca)^{n-1} a$	?	$a^* + b^*$
LEK	$c + cb(ba)^* a$	?	$a^* + b^*$
BLK	$(b + b^2)a^*$	$(a^2)^*(b+c)$	$a^* + b^*$
BLEK	$c + cb(ba)^* a$	?	$a^* + b^*$
AK	✓	?	$a^* + b^*$
AEK	✓	?	$a^* + b^*$
BAK	?	$w(c+d)\widetilde{w}$	$a^n ca^n \cup b^n cb^n$
BAEK	?	?	$a^n ca^n \cup b^n cb^n$

EK $\not\subset$ BK

Lemma

$L = (a^2)^*(b + b^2)$ *is recognized by* $\{$`aa`, `b`$\blacksquare$, `bb`$\blacksquare\}$ *and is not in* BK.

Proof.

If $\mathcal{L}(K) = L$, there exists τ (of normal form $\leftarrow^k b^2$) writing b^2. We distinguish cases according to the value of k.

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If $\mathcal{L}(K) = L$, there exists τ (of normal form $\leftarrow^k b^2$) writing b^2. We distinguish cases according to the value of k.

If $k = 0$, then $\tau \sim b^2$: we then have

$$\varepsilon \cdot \tau \cdot \tau = b^2 \cdot \tau = b^4 \in L.$$

Contradiction

$\square$

EK $\not\subset$ BK

Lemma

$L = (a^2)^*(b + b^2)$ is recognized by $\{\texttt{aa}, \texttt{b}\blacksquare, \texttt{bb}\blacksquare\}$ and is not in BK.

Proof.

If $\mathcal{L}(K) = L$, there exists τ (of normal form $\leftarrow^k b^2$) writing b^2. We distinguish cases according to the value of k.

If $k = 1$, then $\tau \sim \leftarrow b^2$: we then have

$$\varepsilon \cdot \tau \cdot \tau = b^2 \cdot \tau = b^3 \in L.$$

Contradiction

$\square$

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Lemma

$L = (a^2)^*(b + b^2)$ *is recognized by* $\{\texttt{aa}, \texttt{b}\blacksquare, \texttt{bb}\blacksquare\}$ *and is not in* BK.

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If $\mathcal{L}(K) = L$, there exists τ (of normal form $\leftarrow^k b^2$) writing b^2. We distinguish cases according to the value of k.

If $k > 1$ and ***k* even**: from $a^{2k}b \in L$ we obtain

$$a^{2k}b \cdot \tau = a^{k+1}b^2 \in L.$$

Contradiction

$\square$

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If $\mathcal{L}(K) = L$, there exists τ (of normal form $\leftarrow^k b^2$) writing b^2. We distinguish cases according to the value of k.

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Contradiction

□

Research goes on

Decision problems

The membership problem

$$\text{Membership} : \begin{cases} \textsc{Input} : & K \in \mathsf{BAEK}, w \in A^* \\ \textsc{Output} : & w \in \mathcal{L}(K)? \end{cases}$$

- BEK: $\in$ PTIME.
- BLEK: $\in$ PTIME.
- AEK: $\in$ NP.
- BAEK?

Can we do better?

Decision problems

Universality problem

$$\text{Universality} : \begin{cases} \textsc{Input}: & K \in \mathsf{BAEK} \\ \textsc{Output}: & \mathcal{L}(K) = A^*? \end{cases}$$

- BEK: $\in$ PSPACE
- BLEK?
- AEK?
- BAEK?

Other questions?

- Do we have $\mathsf{BEK} \subset \mathsf{BAK}$? $\mathsf{EK} \subset \mathsf{BAK}$?

Other questions?

- Do we have BEK $\subset$ BAK? EK $\subset$ BAK?
- Are all rational languages in BAEK?

$a^* + b^*$ seems to not be in BAEK!

Other questions?

- Do we have $\mathsf{BEK} \subset \mathsf{BAK}$? $\mathsf{EK} \subset \mathsf{BAK}$?
- Are all rational languages in BAEK?
- Is BAEK included in context-sensitive languages? Context-free ones?

Study the keyboard $\{a\blacktriangleright\blacktriangleright, b\blacktriangleleft\blacktriangleleft\}$.

Other questions?

- Do we have $\mathsf{BEK} \subset \mathsf{BAK}$? $\mathsf{EK} \subset \mathsf{BAK}$?
- Are all rational languages in BAEK?
- Is BAEK included in context-sensitive languages? Context-free ones?
- Relations to other known models?

Thanks for your attention!

Thanks for your attention!

Questions?

An example

$$K_C = \{\leftarrow\texttt{a}\lozenge\blacklozenge, \leftarrow\leftarrow\texttt{b}\lozenge\blacklozenge\blacklozenge\}.$$

An example

Some as and bs separated by $\lozenge$ and $\blacklozenge$.

- Between two a: $\lozenge$.
- Between two b: $\lozenge$.
- Between an a and a b: nothing.
- Between a b and an a: $\lozenge\blacklozenge$.

$$K_C = \{\leftarrow\mathtt{a}\lozenge\blacklozenge, \leftarrow\leftarrow\mathtt{b}\lozenge\blacklozenge\blacklozenge\}.$$

An example

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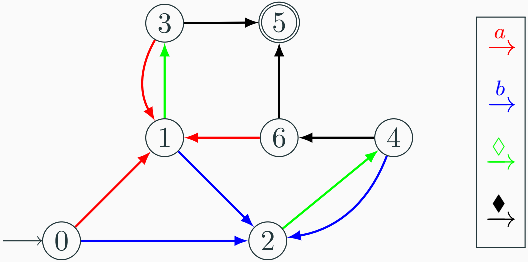

An example

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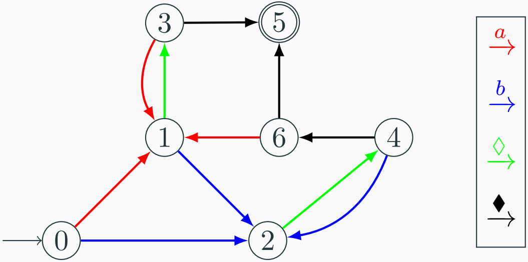
$$K_C = \{\leftarrow\mathtt{a}\lozenge\blacklozenge, \leftarrow\leftarrow\mathtt{b}\lozenge\blacklozenge\blacklozenge\}.$$

$$(b(\lozenge b)^*\lozenge\blacklozenge + (a + b(\lozenge b)^*\lozenge\blacklozenge a)((\lozenge + b(\lozenge b)^*\lozenge\blacklozenge)a)^*(\lozenge + b(\lozenge b)^*\lozenge\blacklozenge))\blacklozenge$$



Class inclusions

Lemma ($\mathsf{LK} \not\subset \mathsf{BEK}$)

The language of even palindromes is in LK *via* $\{aa\blacktriangleleft, bb\blacktriangleleft\}$*, and is not rational.*

Lemma ($\mathsf{BK} \not\subset \mathsf{AK}$ and $\mathsf{EK} \not\subset \mathsf{AK}$)

Finite languages are in EK *and* BK*, but not* AK.

BAK $\not\subseteq$ AEK

Lemma

$L = a^* + b^* \notin$ AEK.

Proof.

- There is a (non-final) key writing an a.
- There is a (non-final) key writing a b.

We can write a word with a and b! $\square$

Rat $\not\subset$ BEK

Lemma

$a^*b^* \notin$ BEK

Proof.

- There exists τ writing a and applying entry (τ is of the form $\leftarrow^k a\blacksquare$).
- There exists τ' writing arbitrarily many b without entry (for instance $k+1$).

$\tau'\tau$ writes ba and ends the execution. $\square$