On Finite Monoids over Nonnegative Integer Matrices and Short Killing Words

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February 17th 2020

Matrix mortality

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No product of As and Bs can be zero, but

$$CBA = C(BA) = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Given a finite set of matrices $S = \{M_1, ..., M_p\} \subseteq \mathcal{M}_{n\times n}(\mathbb{Z})$, the problem of deciding if the monoid generated by S contains 0 is

- Undecidable in the general case (Paterson, 1970).

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- Polynomial-time when all entries are non-negative and the monoid generated by S is finite :
  One can compute the average matrix, and check if its spectral radius is less than 1, but this does not provide a witness.
  $\rightarrow$ Can one compute a 'short' sequence of matrices $M_1, ..., M_p$ such that $M_1 ... M_p = 0$?

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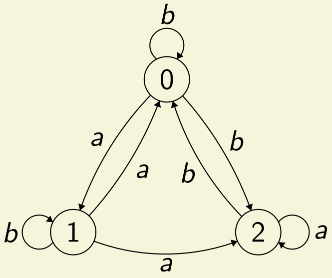
$\rightarrow$ Can one compute a 'short' sequence of matrices $M_{i_1}, ..., M_{i_k}$ such that $M_{i_1}...M_{i_k} = 0$?

Matrix mortality

We can switch from a problem on matrices to a problem on automata :



$\Leftrightarrow$

$$M(a) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

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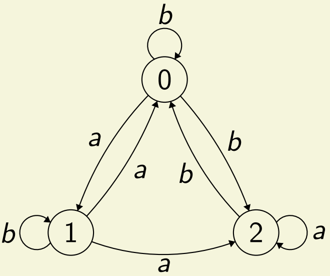

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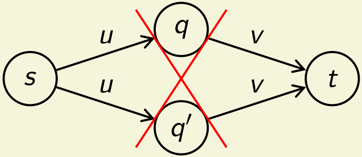
The *rank* of a word w in $\mathcal{A}$ is the rank of $M(w)$.
A *killing word* is a word of rank 0.

Unambiguous automata

Unambiguous automaton $\rightarrow$ Nondeterministic finite automaton in which for all states s, t, for all word w there is at most one path from s to t labelled by w.
Equivalently, all entries of the matrix monoid associated to this automaton are 0 or 1.



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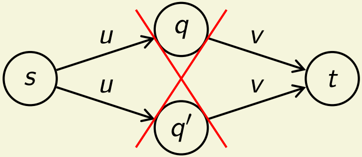
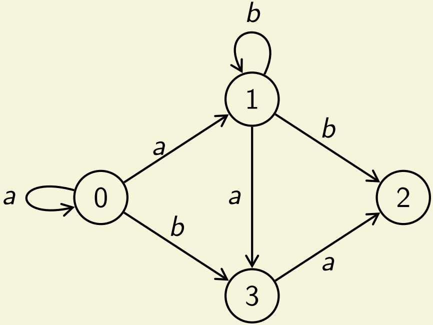


The monoid associated to an unambiguous automaton is finite. The converse is true for strongly connected automata.

Unambiguous automata



$$M(a) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; M(b) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M(baab) = 0$$

Result

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Given an unambiguous automaton with n states one can **compute** a killing word (when there is one) of length at most $\frac{1}{16}n^5 + \frac{15}{16}n^4$ in polynomial time.

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Main result

Given a set of matrices $S \subseteq \mathcal{M}_{n\times n}(\mathbb{N})$ generating a finite monoid, if this monoid contains 0,then one can **compute** a sequence of matrices $M_1, ..., M_K \in S$ such that $M_1 ... M_K = 0$ in polynomial time, with $K \leq \frac{1}{16}n^5 + \frac{15}{16}n^4$.

When the automaton does not have any killing word, the procedure returns a word of minimal rank.

Result

Main result (generalized version)

Given a set of matrices $S \subseteq \mathcal{M}_{n\times n}(\mathbb{N})$ generating a finite monoid, one can **compute** in polynomial time a sequence of matrices $M_1, ..., M_K \in S$ such that $M_1 ... M_K$ has minimal rank, with $K \leq \frac{1}{16}n^5 + \frac{15}{16}n^4$.

Remarks

- In 1988, Carpi gave a polynomial-time algorithm to compute minimal-rank words for strongly connected unambiguous automata with positive minimal rank.

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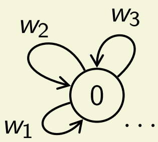
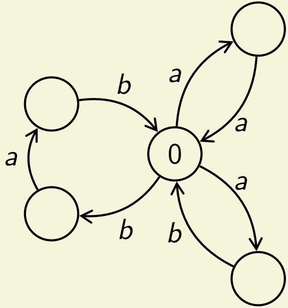
- The main contribution here is the extension of Carpi's result to unambiguous automata with minimal rank 0.

Restivo's conjecture

$S = \{w_1, ..., w_p\} \subseteq \Sigma^*$



$S = \{aa, ab, bab\}$



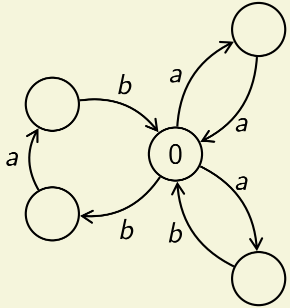
Let $\mathcal{A}(S)$ be the automaton associated to the set of words S.
$\mathcal{A}(S)$ is non-deterministic in general.

Restivo's conjecture

Restivo's conjecture (1981)

$kw(\mathcal{A}(S))$ is bounded by $2m^2$ where $m = max_{w \in S}|w|$

Here $\mathcal{A}(S)$ is not necessarily unambiguous.



$S = \{aa, ab, bab\}$

The word *bbb* is a killing word for this automaton.

History

1981 $\rightarrow$ Restivo's conjecture ($2m^2$ upper bound)

2010 $\rightarrow$ Numerical counterexample by Fici, Pribavnika and Sakarovitch

2011 $\rightarrow$ Family of counterexamples by Gusev and Pribavnika ($5m^2$ lower bound)

2017 $\rightarrow$ Computations hinting at an exponential growth in m by Julia, Malapert and Provillard

2019 $\rightarrow$ Superpolynomial lower bound by Mika and Szykuła ($2^{\frac{m}{4}}$)

Final disproval

Theorem (Maksymilian Mika, Marek Szykuła)

The following problem is PSPACE-complete:
Input: A finite set S of words
Output: $\Sigma^* = S^*$?

While proving this result the authors constructed a family of sets of words whose minimal uncompletable word is of superpolynomial size in the maximal length of their elements.

A particular case

Rather than any finite set of words, we can restrict ourselves to finite codes.

Definition

A code is a set of words S such that for all $u_1, \ldots, u_n, v_1, \ldots, v_p \in S$, if $u_1 \cdots u_n = v_1 \cdots v_p$ then $n = p$ and $u_i = v_i$ for all i.

$S = \{aa, aba\}$ is a code.

A particular case

When S is a finite code, $\mathcal{A}(S)$ is unambiguous (the converse is true).

A version of Restivo's conjecture in codes

When S is a code, $kw(\mathcal{A}(S))$ is polynomially bounded in terms of $m = max_{w \in S}|w|$

This has been shown for some particular cases such as prefix codes (Néraud, Selmi, 1988).

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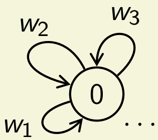
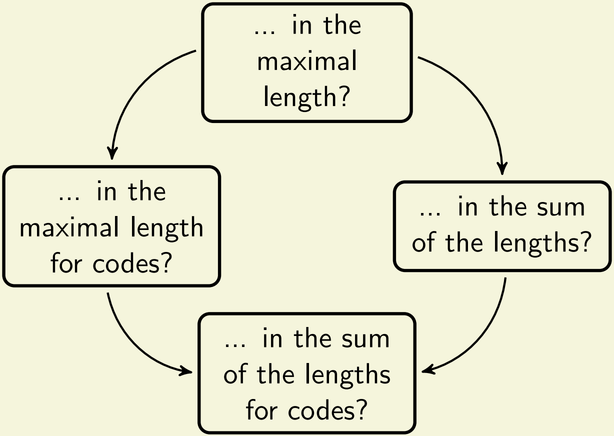
Weaker version of the conjecture

When S is a code, $kw(\mathcal{A}(S))$ is polynomially bounded in terms of the number of states n in $\mathcal{A}(S)$



Is there always a killing word of length polynomial in ...



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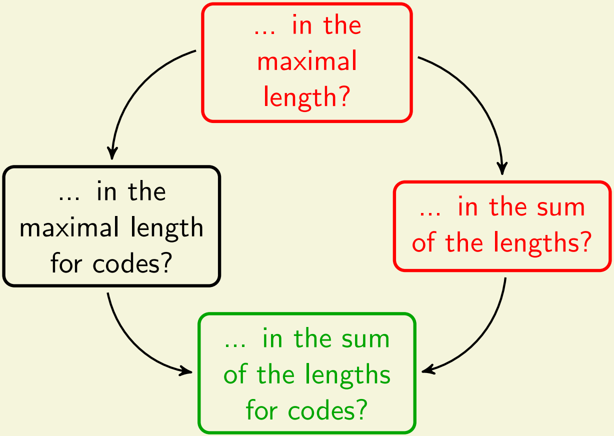
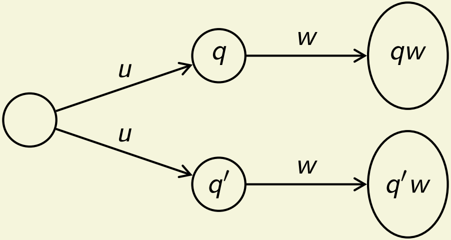


Proof for strongly connected UFA



qw and $q'w$ are disjoint because of unambiguity.

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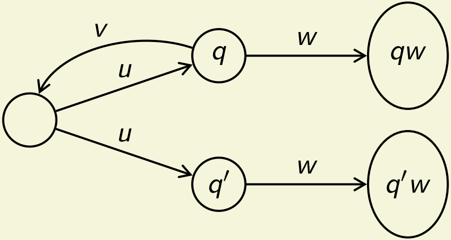

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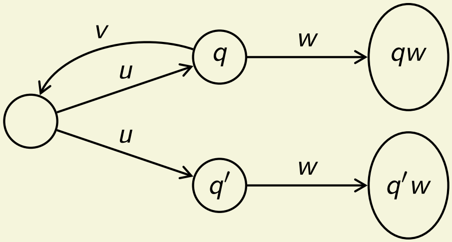

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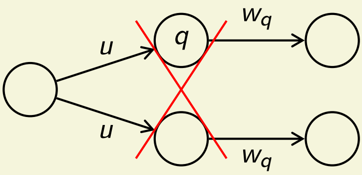
$qw \sqcup q'w \subseteq q(vuw)$

Proof for strongly connected UFA

Lemma

For all state q we can compute a word w_q such that for all states $q' \neq q$, if q and q' are coaccessible then either $qw = \emptyset$ or $q'w = \emptyset$.

For all q there exists w_q such that



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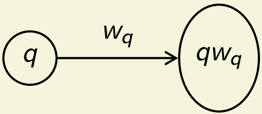

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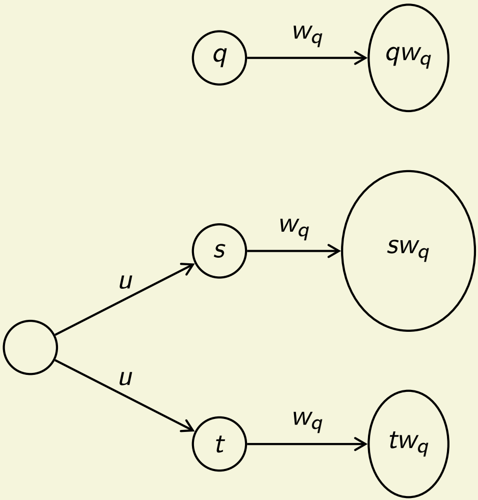

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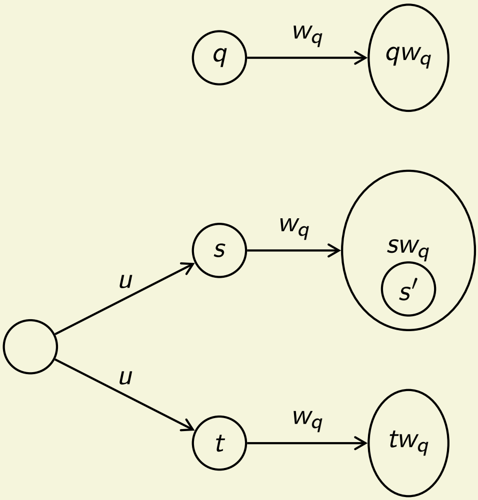

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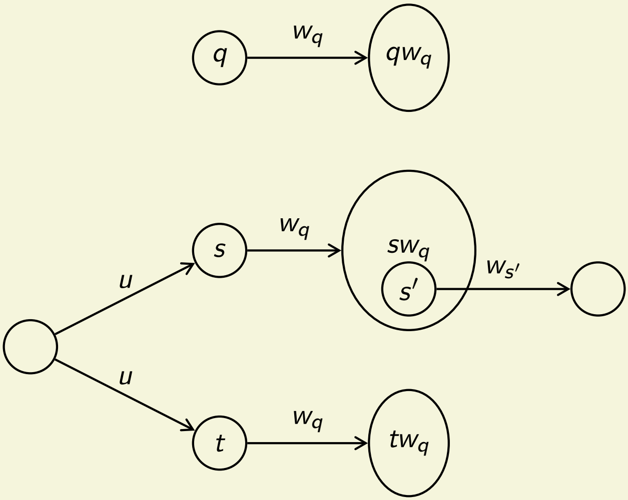

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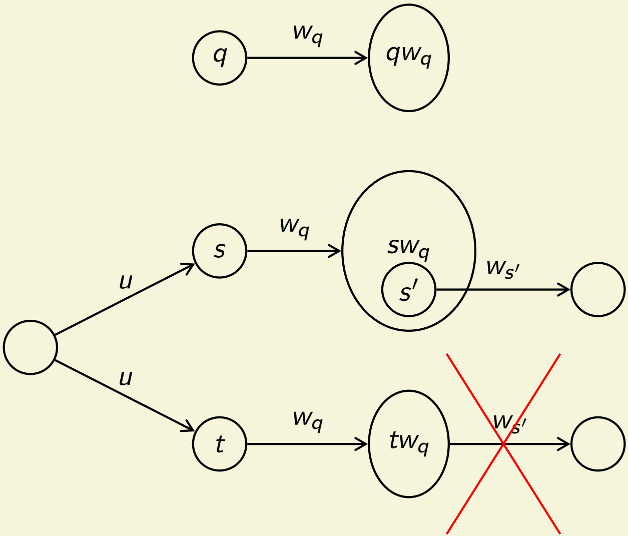

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We can compute a word w such that for all states $q \neq q'$, if q and q' are coaccessible then either $\delta(q, w) = \emptyset$ or $\delta(q', w) = \emptyset$.

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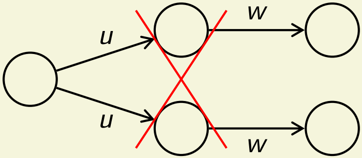

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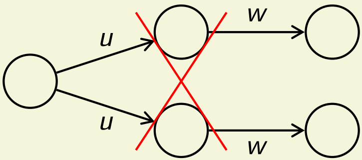
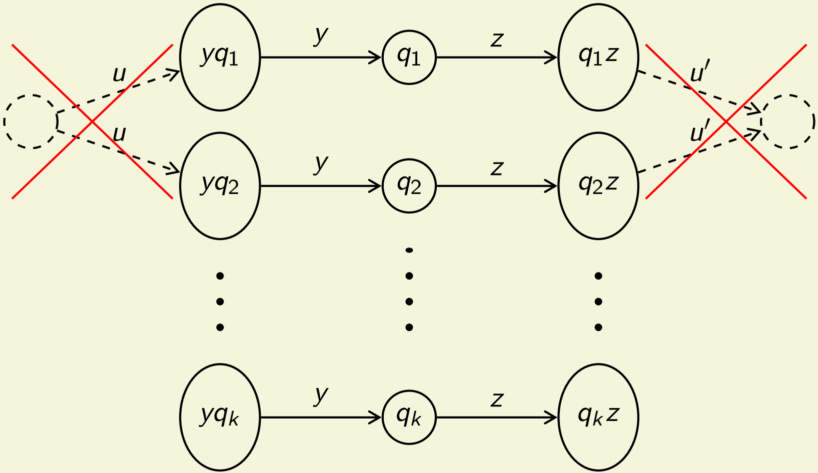


Idea of the proof



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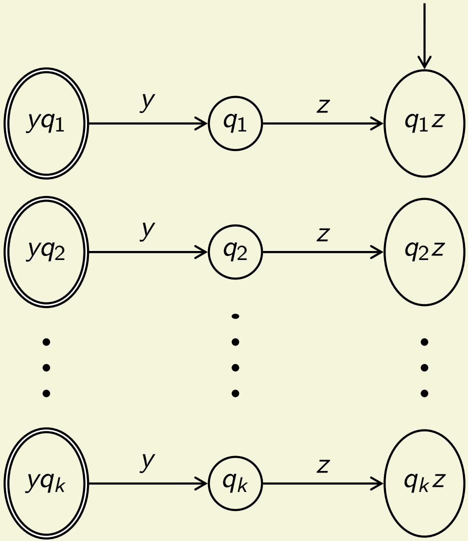
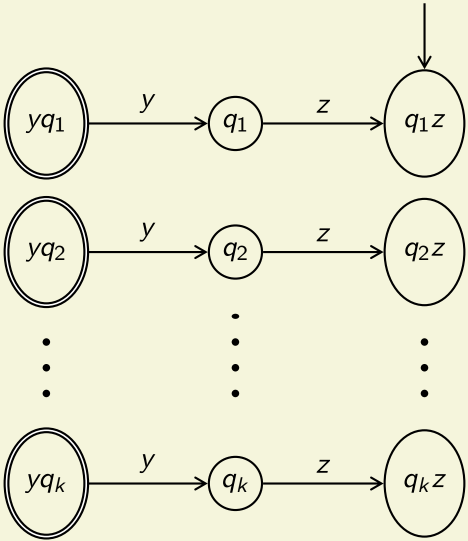


Proof for strongly connected UFA

Lemma

Given a non universal automaton $\mathcal{A}$ with n states such that any word has at most one accepting run, one can compute in polynomial time a word $w \notin L(\mathcal{A})$, with $|w| \leq n$.

Idea of the proof



A counter-example

Main result (again)

Given an unambiguous automaton $\mathcal{A}$, one can compute a killing word w in polynomial time, if it exists, with $|w| \leq \frac{1}{16}n^5 + \frac{15}{16}n^4$

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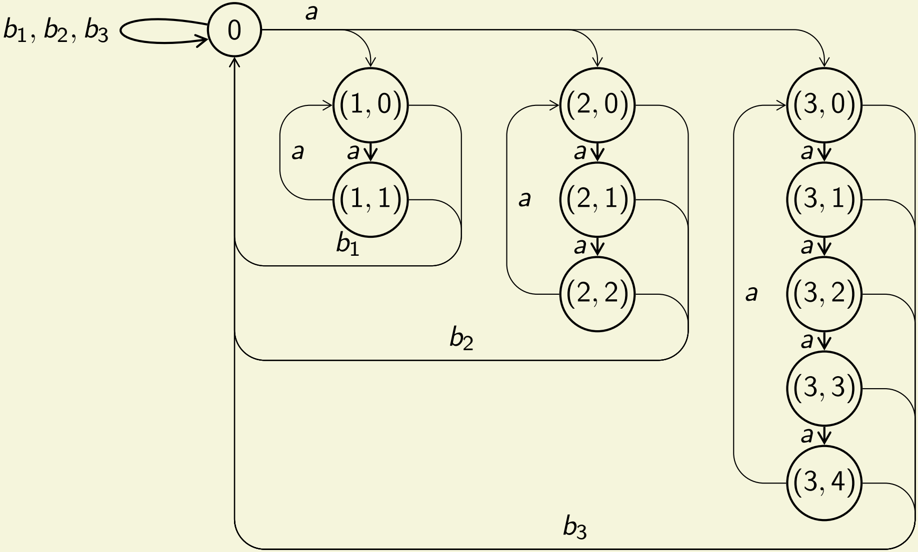
Are all minimal-rank matrices products of polynomially many matrices of S ?

A counter-example



Conclusion

- Applications in the theory of codes
- Some work to improve the degree of the bound
- Possible extensions to finite monoids of integer matrices for instance.