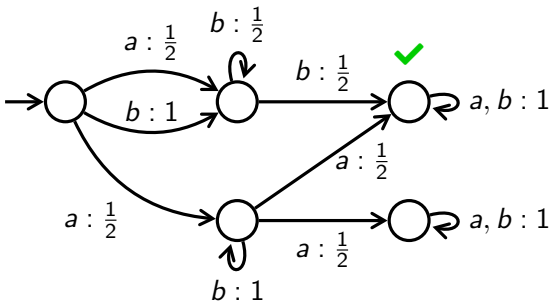
The Short-Ends Factorisation Theorem and applications

Corto Mascle
MPI-SWS Kaiserslautern

Based on joint work with
Hugo Gimbert + Patrick Totzke and Richard Mandel + Georg Zetzsche
LaBRI, Bordeaux University of Liverpool MPI-SWS Kaiserslautern

Controlling a random population



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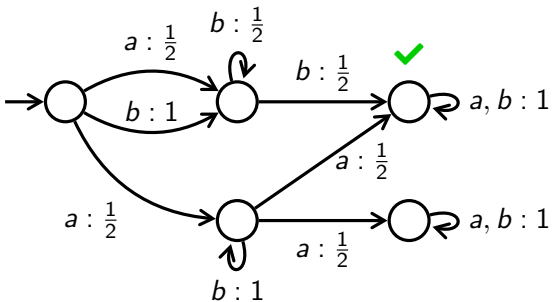

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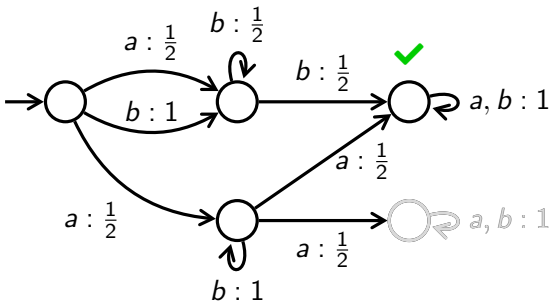

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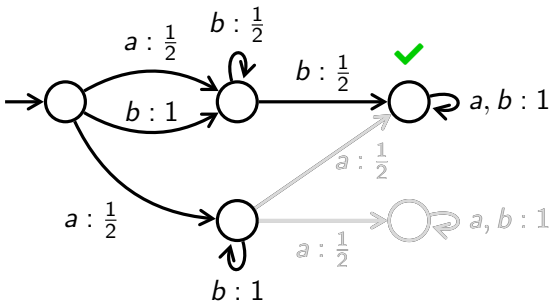

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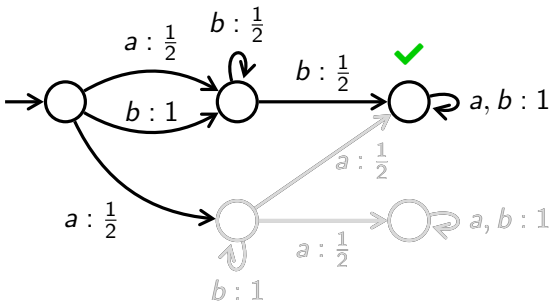

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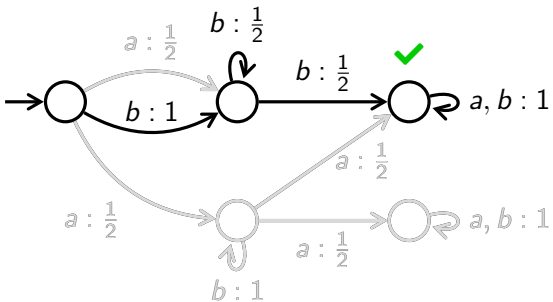

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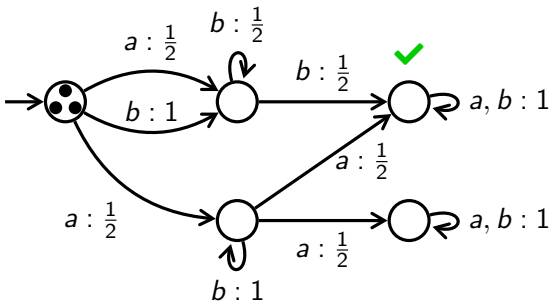

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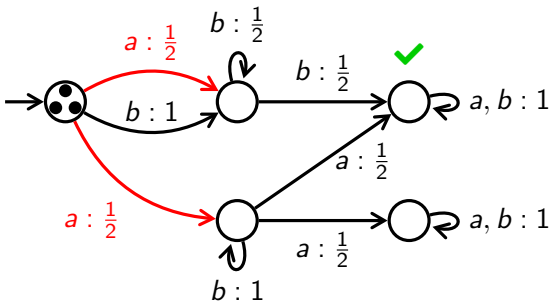

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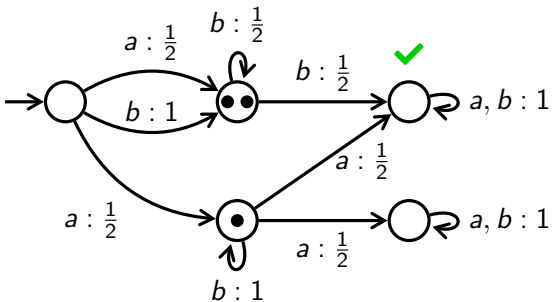

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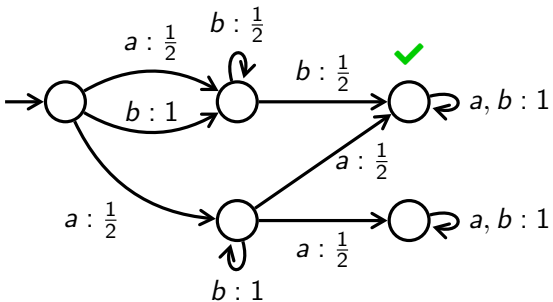

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$\Rightarrow$ Product of N MDPs.

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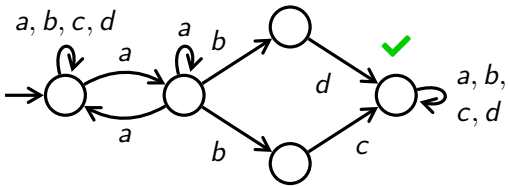

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Given an MDP $\mathcal{M}$, is there a winning strategy against N tokens, for all N?

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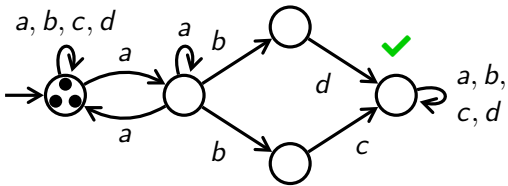

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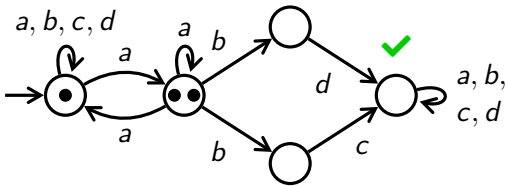

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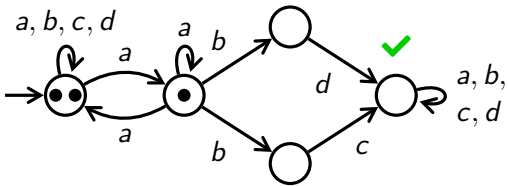

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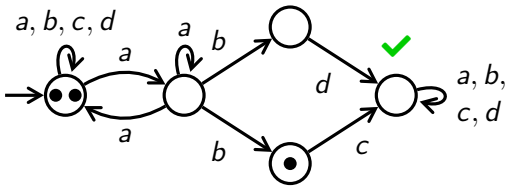

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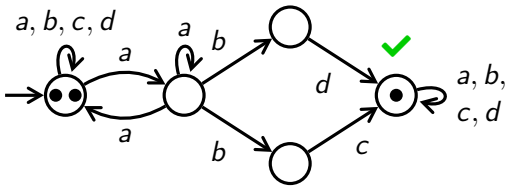

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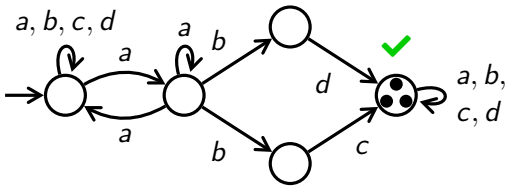

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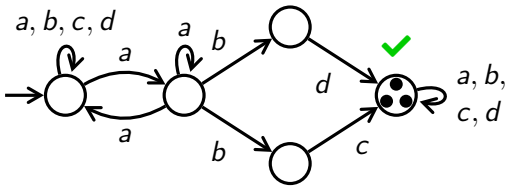

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Adapted from [Bertrand Dewaskar Genest Gimbert '15]

There are MDPs of size k for which we can win against 2^{2^k} tokens and not more.

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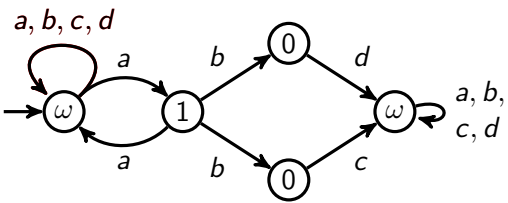

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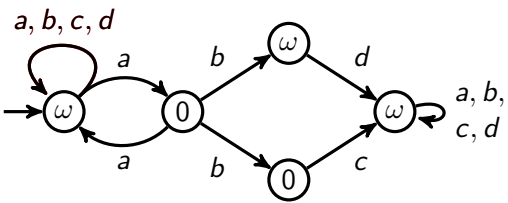

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Given N identical Markov decision processes, make them all reach ✔ with probability 1.

$S \leftarrow \mathbb{N}^Q$ → configurations
$C \leftarrow \mathbb{N}^Q \times \Sigma$ → commits
while not fixpoint **do**
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Put constraints on transitions instead of states!

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A *tile* is a function $Q \times Q \to \mathbb{N} \cup \{\omega\}$
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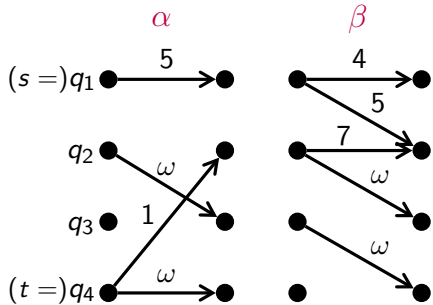

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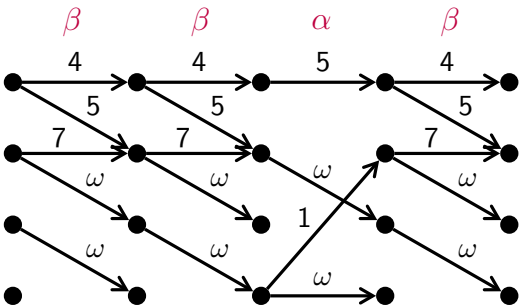

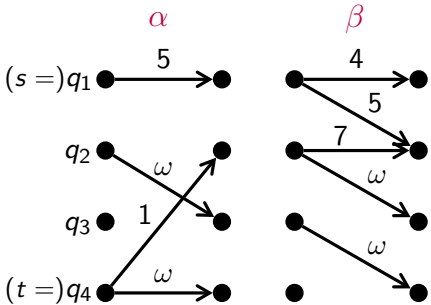

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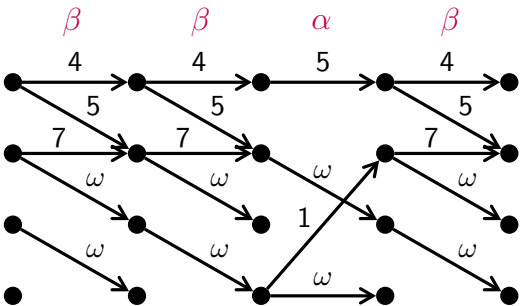

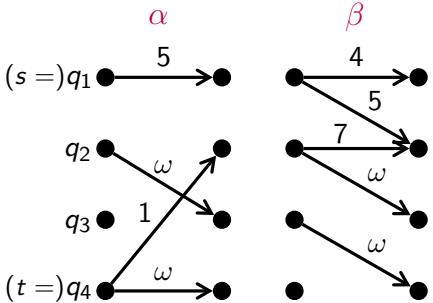

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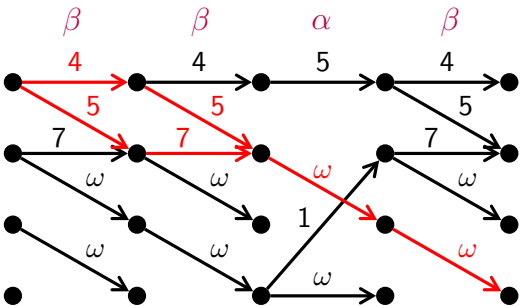

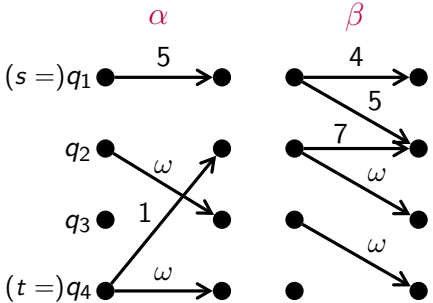

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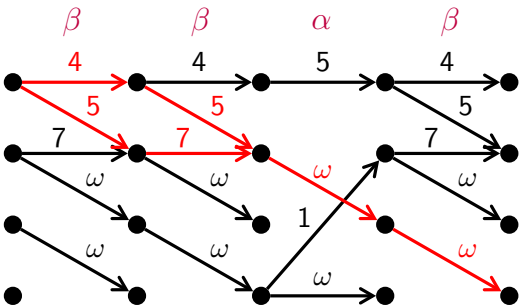

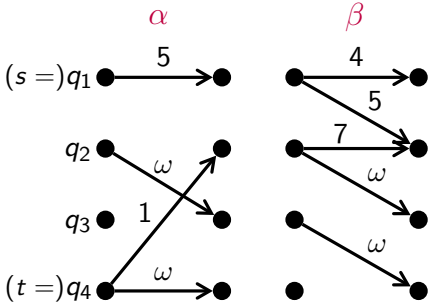

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Input: *A set of tiles* **Tiles**
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[Blumensath Colcombet Parys '16]

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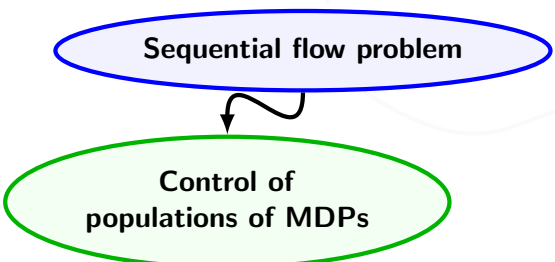
Theorem ([GMT '25])

The Randomised Population Control Problem is EXPTIME-complete.

Summary



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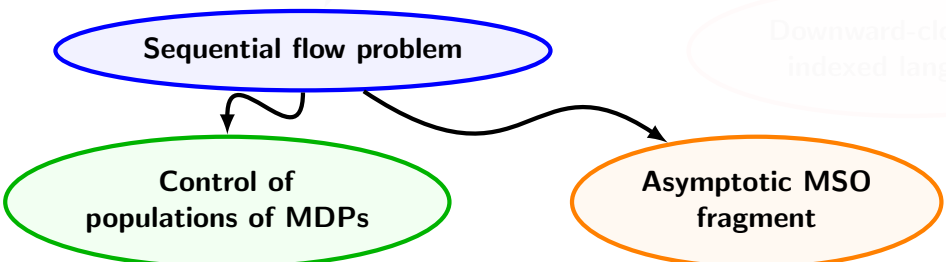
Sequential flow problem
Control of
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Asymptotic MSO
fragment

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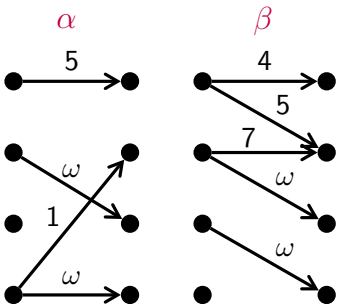

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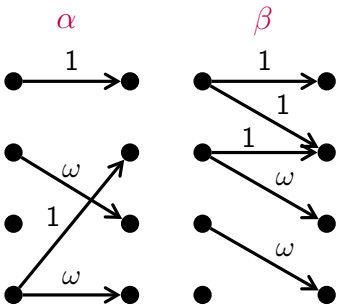

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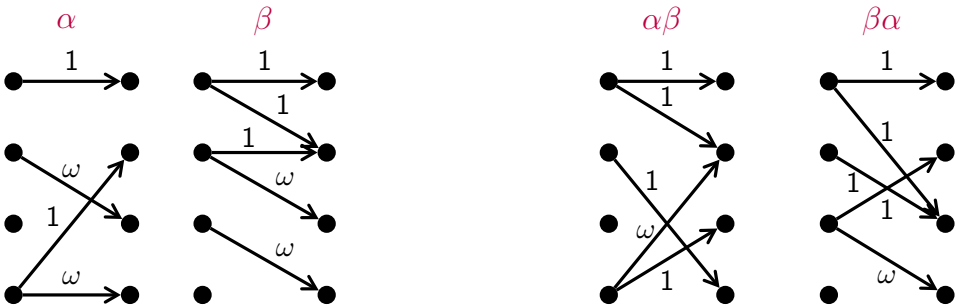

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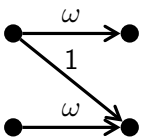

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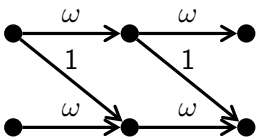

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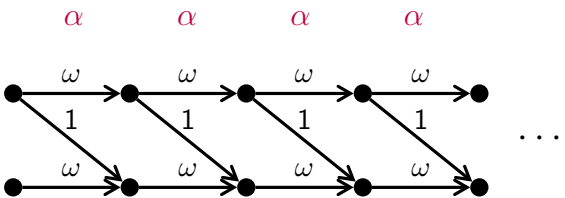

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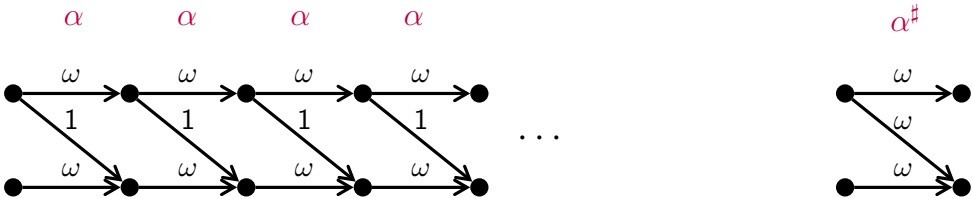

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Define a new operator $\sharp$ on idempotents:

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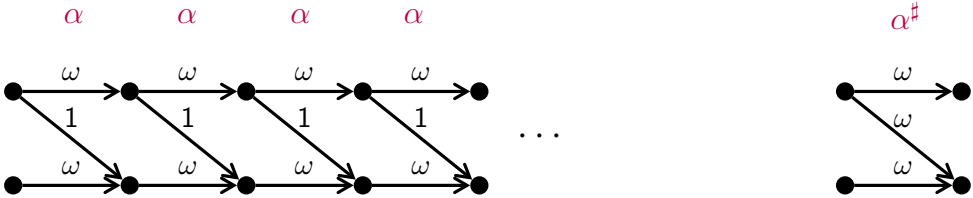

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If $\alpha^\sharp(s,t) = \omega$ then we have unbounded flows between s and t.

Abstraction

$\mathcal{F}_\sharp$ the closure of $\varphi(\textbf{Tiles})$ under product and $\sharp$.

Lemma

$\mathcal{F}_\sharp$ contains some α with $\alpha(s,t) = \omega$ if and only if there are unbounded flows between s and t.

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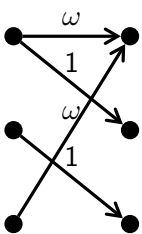
`How large can a bounded flow be?`

Iteration dichotomy

α



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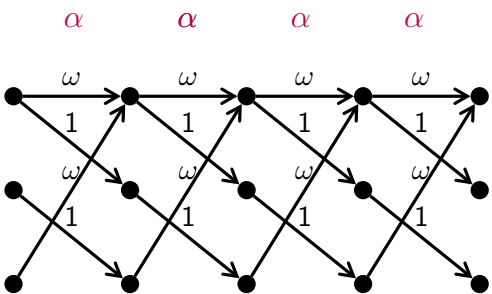

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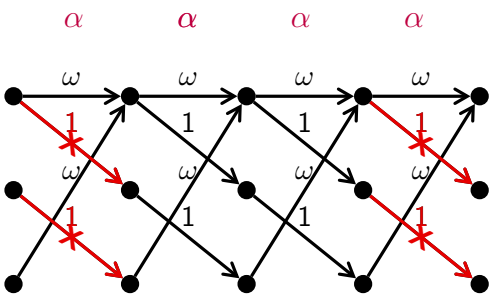

For all idempotent α and $s, t \in Q$,

- either $\alpha^\sharp(s,t) = \omega$
- or there is a cut between s and t in the first and last α.

Short-ends factorisation theorem (based on work by [Simon '90], [Colcombet '11])

Σ finite alphabet, $(\boldsymbol{M}, \cdot)$ finite monoid, $\varphi : \Sigma^* \to \mathbf{M}$ morphism.

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Σ finite alphabet, $(\boldsymbol{M}, \cdot)$ finite monoid, $\varphi : \Sigma^* \to \mathbf{M}$ morphism.
`Factorisation tree:` Node labels in $\Sigma^* \times \boldsymbol{M}$.

3 types of nodes:

Leaves

$(a, \varphi(a))$

Product nodes

$(uv, x \cdot y)$

(u, x) (v, y)

Idempotent nodes

$(u_1 u_2 \cdots u_n, e)$

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`Example on the board` ⟶

Ramsey bounds

$R_{\mathbf{M}}(k)$ = minimal n such that every word w of length n has a factor $u_1 \dots u_k$ with $\varphi(u_1) = \cdots = \varphi(u_k) = e$ an idempotent of $\mathbf{M}$.

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Theorem

For all $w \in \Sigma^*$, there is a factorization tree of height $poly(log(\mathbf{M}), log(R_{\mathbf{M}}(3)))$.

Corollary

In a transition monoid of dimension n, every word has a factorization tree of height $poly(n)$.

Based on bounds by [Jecker '21].

Exponential bound

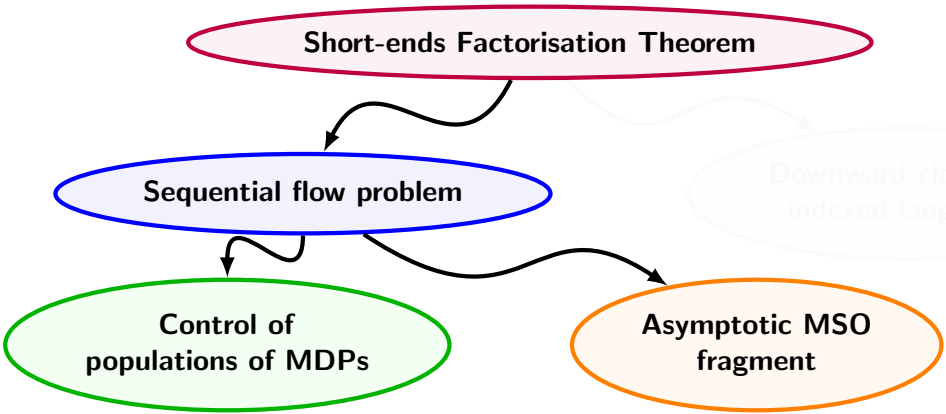
We can show:

- Every word has a factorisation of polynomial height in the flow monoid.
- If there is a bounded cut between s and t in all w, then there is one that lives within the factorisation.

Theorem

If the flow between s and t is bounded then it is at most exponential in $|Q|$.

Summary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Another application: Indexed grammars

Indexed grammar= Context-free grammar where each non-terminal carries a stack.
N set of *non-terminals*, T set of *terminals*, Γ set of *stack symbols*.

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Indexed grammars : Examples

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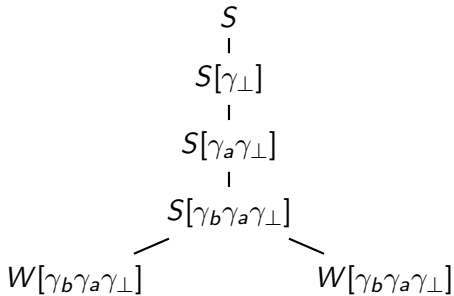

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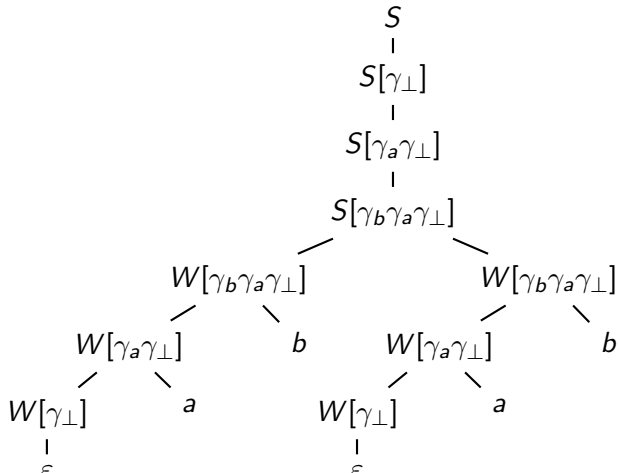

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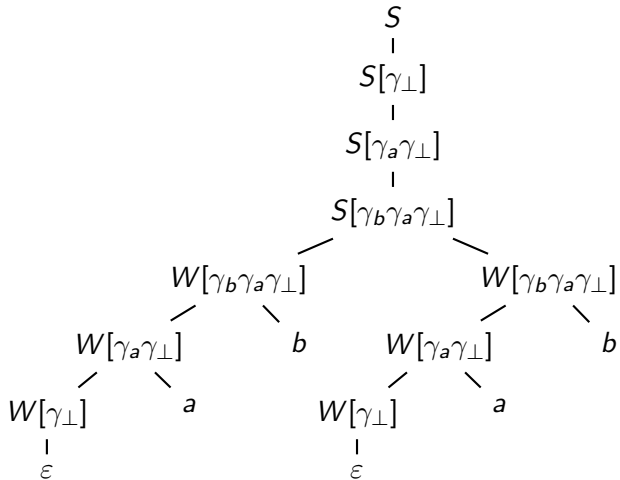

Subword-closures

Subword

w is a **subword** of w' ($w \preceq w'$) if we can remove letters of w' to get w.

Ex: bab is a subword of $a\underline{b}b\underline{a}\underline{b}a$.

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- Verification of thread pools with bounded context switching
- Separation by piecewise-testable languages
- Lossy channel machines

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[Courcelles '91]

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Subword-closure for indexed grammars

Given an indexed grammar $\mathcal{I}$ of size k, how large can an automaton recognising $L(\mathcal{I})\downarrow$ be?

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A stack content $z \in \Gamma^*$ is mapped to a boolean matrix $M(z) \in \{\top, \bot\}^{N \times N}$.

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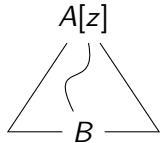

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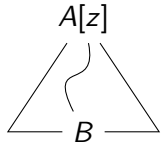
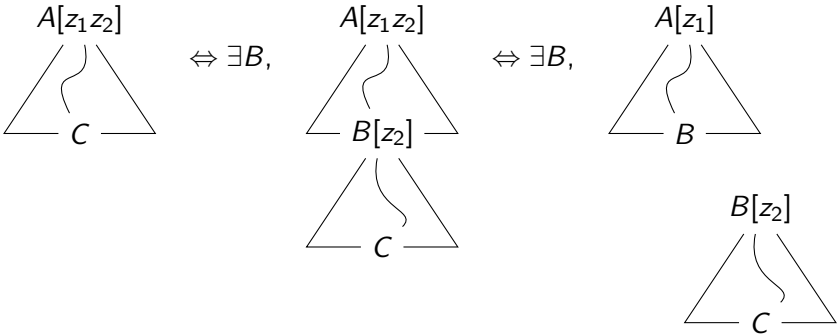


M is a morphism: $M(z_1 z_2) = M(z_1)M(z_2)$ because



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Replace stacks with their short-ends factorization in the production monoid!

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The short-ends factorization of γz is determined by the one of z.

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For each γ define $\mathtt{push}_\gamma$ the function mapping the factorization of z to the factorization of γz and $\mathtt{pop}_\gamma = \mathtt{push}_\gamma^{-1}$.

Abstraction

Define a context-free grammar with non-terminals $A[\sigma]$,

- $A \in N$
- σ short-ends factorisation

$$
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\quad \Rightarrow \quad
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\end{aligned}
$$

Theorem

This CFG has the same subword-closure as the indexed grammar.

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Extensions to higher-order pushdown automata...

Summary

Short-ends Factorisation Theorem

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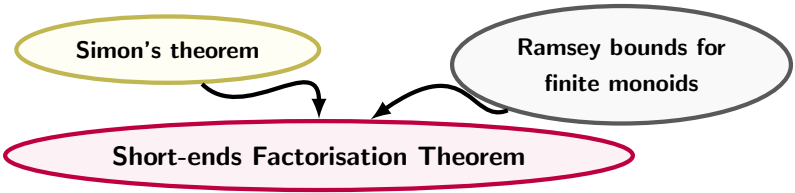
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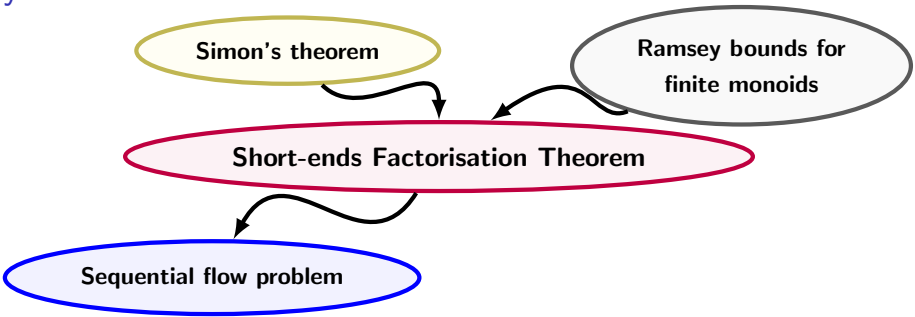
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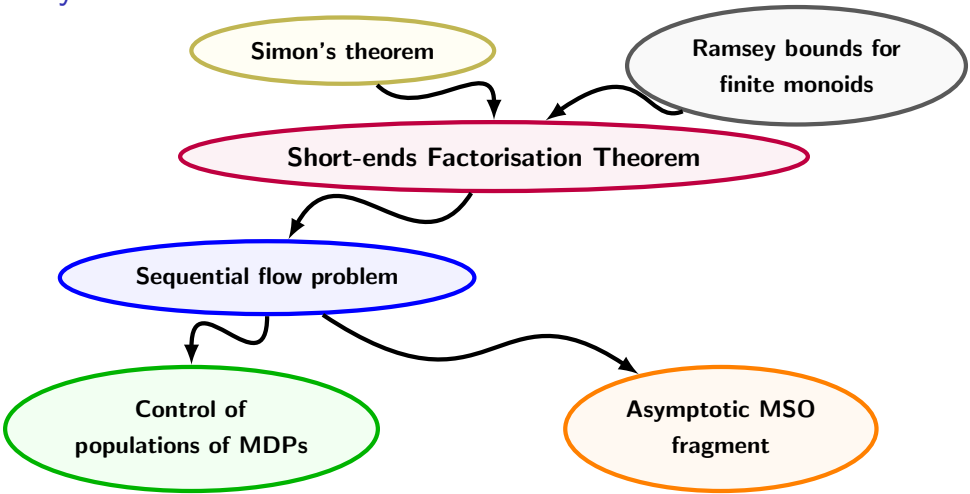
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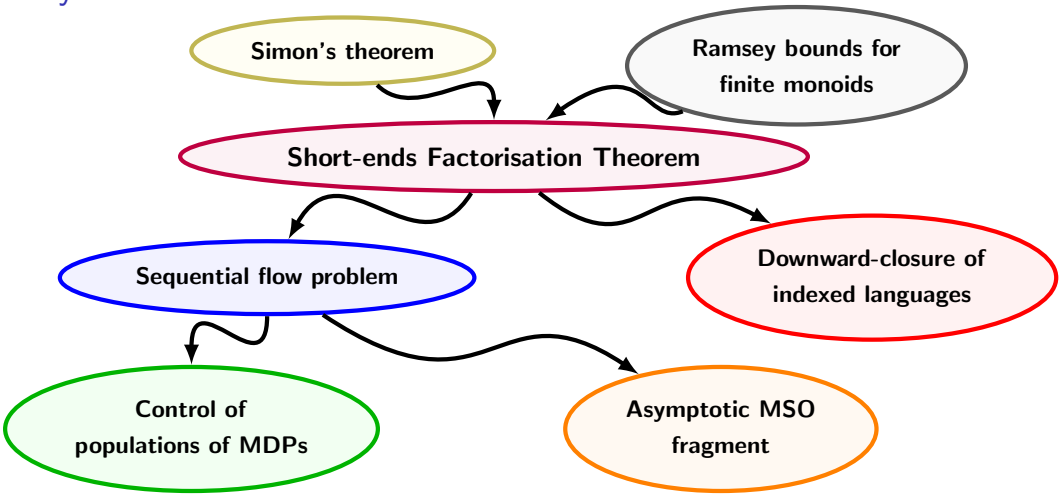
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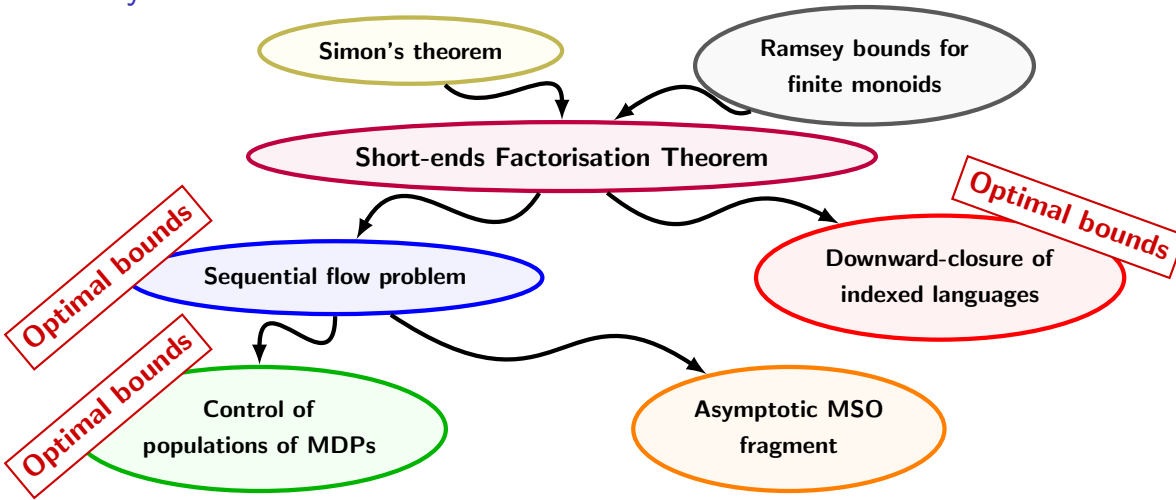
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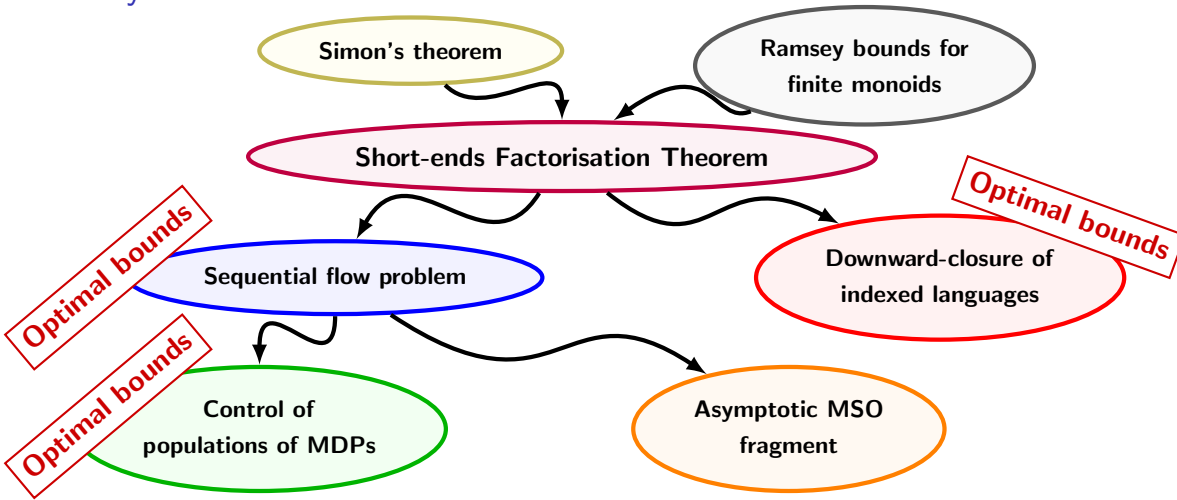

Thanks!