Optimal sequential flows and a short-ends factorisation theorem

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Sequential Flow Problem [Colcombet, Fijalkow, Ohlmann '20]

V a set of vertices.
A *tile* is a function $V \times V \to \mathbb{N} \cup \{\omega\}$
describing *capacities*.

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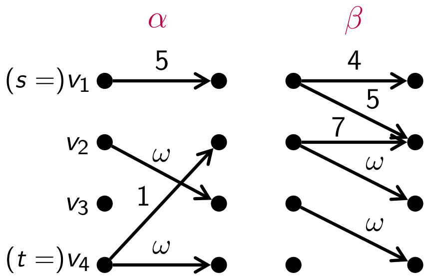

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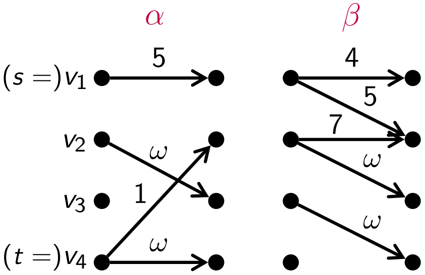

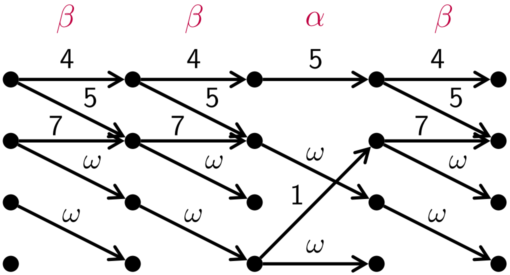

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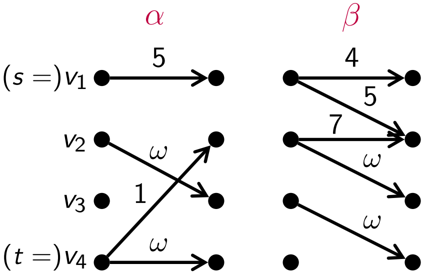
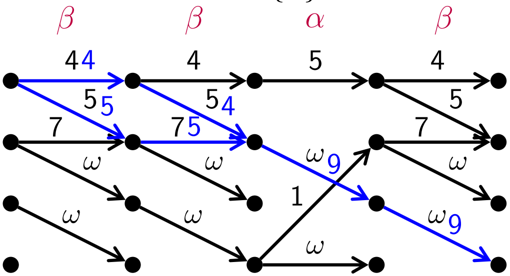


$\texttt{MaxFlow}$: $\textbf{Tiles}^* \to \mathbb{N} \cup \{\omega\}$



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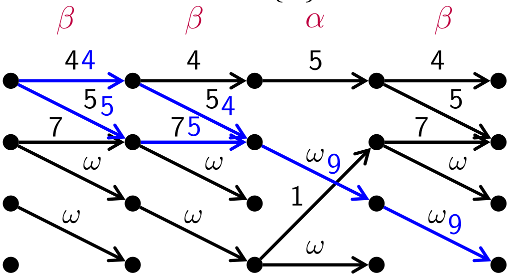

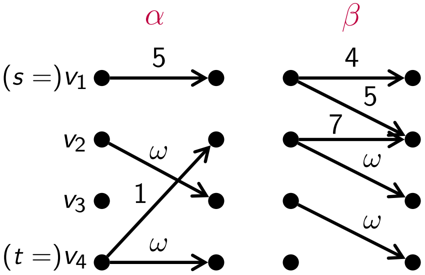

Problem

Input: A set of tiles **Tiles**
Output: Is $\{\texttt{MaxFlow}(w) \mid w \in \textbf{Tiles}^*\}$ unbounded?

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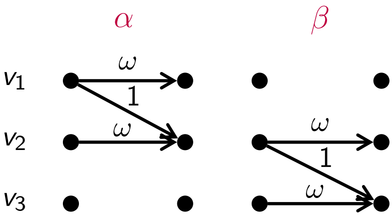
Example



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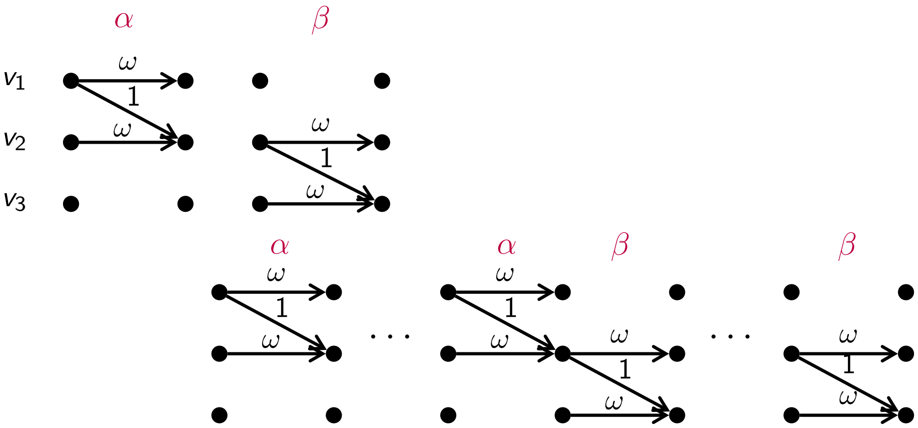
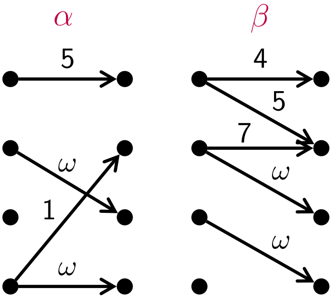


α
β
v_1
v_2
v_3
ω
1
ω
ω
1
ω
α
α
β
β
$\cdots$
$\cdots$

Abstraction



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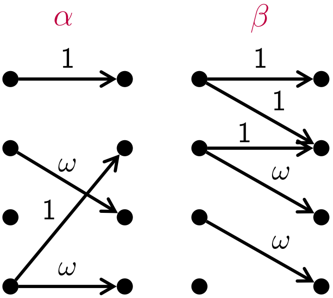

Morphism φ : **Tiles*** $\to \{0, 1, \omega\}^{V \times V}$ with max-min product.

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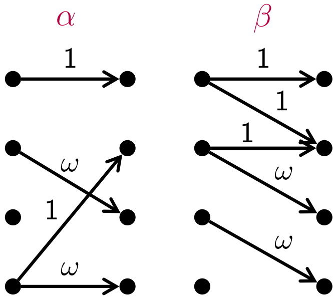

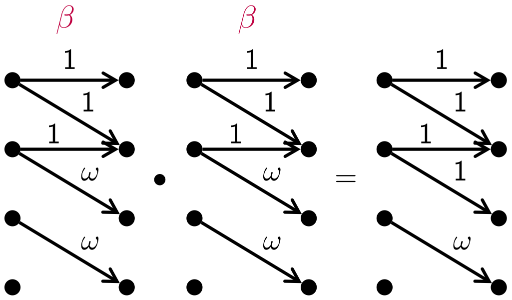

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Iteration (Inspired by work of Simon, Hashigushi, Leung)

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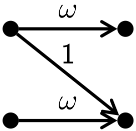

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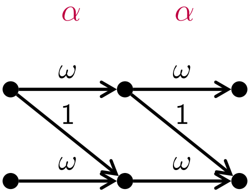

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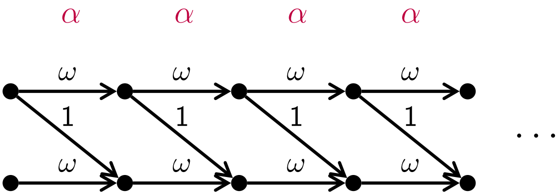

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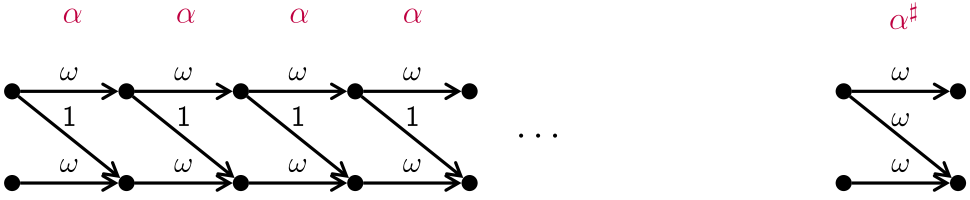

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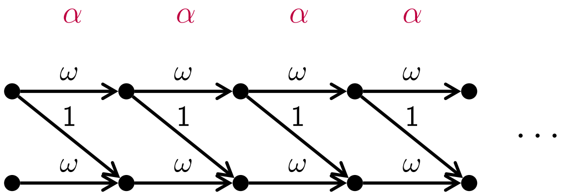

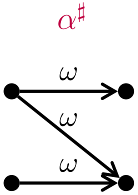

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If $\alpha^\sharp(s, t) = \omega$ then we have unbounded flows between s and t.

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What about the other direction?

Factorisation forest theorem [Simon '90]

Σ finite alphabet, $(\boldsymbol{M}, \cdot)$ finite monoid, $\varphi : \Sigma^* \to \mathbf{M}$ morphism.

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Factorisation tree: Node labels in $\Sigma^* \times \boldsymbol{M}$.

3 types of nodes:

Leaves

$(a, \varphi(a))$

Product nodes

$$(uv, x \cdot y)$$
$$(u, x) \quad (v, y)$$

Idempotent nodes

$$(u_1 u_2 \cdots u_n, e)$$
$$(u_1, e) \quad \cdots \quad (u_n, e)$$

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Theorem

Every word has a factorisation tree of height $\leq 3|\mathbf{M}|$

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Proof by induction on the height of factorization trees.

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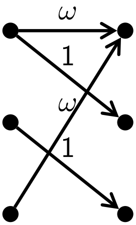
Proof by induction on the height of factorization trees.

But how large can a bounded flow be in terms of nb of vertices n?

Using Simon's theorem $\rightarrow 2^{3|\mathcal{F}_\#|} \leq 2^{3^{n^2+1}}$
Can we do better?

Iteration dichotomy

α



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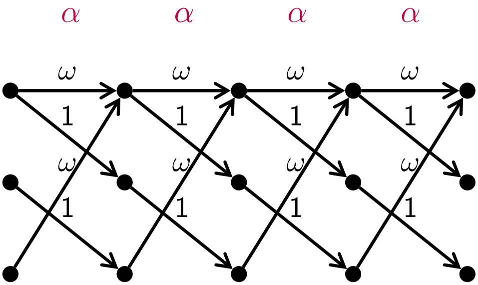

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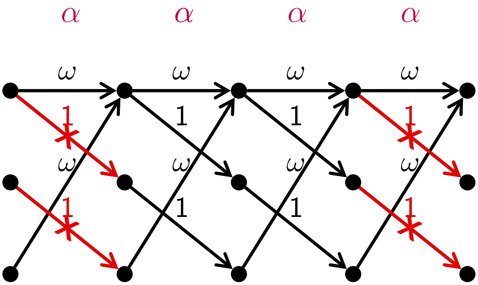

For all idempotent α and $s, t \in Q$,

- either $\alpha^\sharp(s,t) = \omega$
- or there is a bounded *cut* between s and t in the first and last α.

Short-ends factorisation theorem (based on work by [Simon '90], [Colcombet '11])

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$\varphi(u_i) = e$ for all i.

Ramsey bounds (based on work by [Jecker '21])

$R_{\mathbf{M}}(k) =$ minimal n such that every word w of length n has a factor $u_1 \dots u_k$ with $\varphi(u_1) = \cdots = \varphi(u_k) = e$ an idempotent of $\mathbf{M}$.

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Theorem

For all $w \in \Sigma^*$, there is a factorization tree of height $poly(log(\mathbf{M}), log(R_{\mathbf{M}}(3)))$.

Corollary

In a transition monoid $\{\top, \bot\}^{n \times n}$, every word has a factorization tree of height $poly(n)$.

Exponential bound

- Every word has a factorisation of polynomial height in n.
- If there is a bounded cut between s and t in a sequence of tiles, then there is one that lives within the factorisation.

Theorem

If the flow between s and t is bounded then it is at most exponential in n.

Applications

Short-ends Factorisation Theorem

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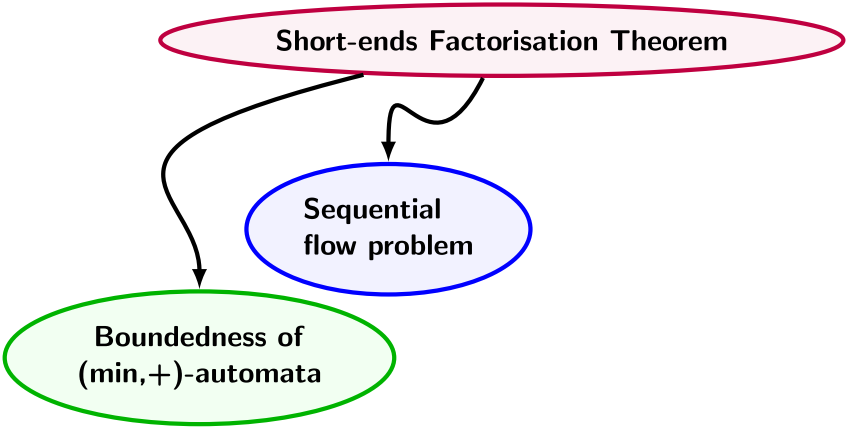
Short-ends Factorisation Theorem

Boundedness of (min,+)-automata

Applications



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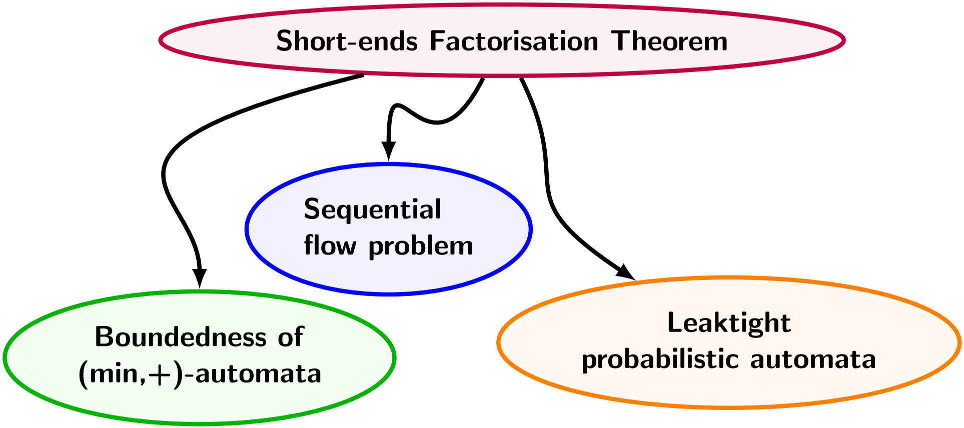

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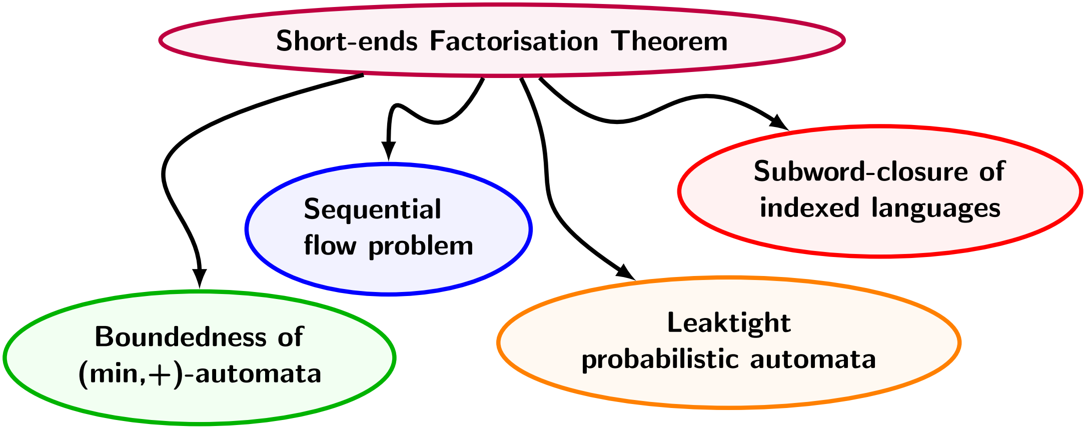

Theorem

Let $\varphi : \Sigma^* \to \mathbf{M}$ *a morphism.*
Every $w \in \Sigma^*$ *has a short-ends factorization tree of height* $poly(log(\mathbf{M}), R_{\mathbf{M}}(3))$.

Corollary

Let $\mathbb{M}_n = \{\top, \bot\}^{n \times n}$ *and* $\varphi : \Sigma^* \to \mathbf{M}$ *a morphism.*
Every $w \in \Sigma^*$ *has a short-ends factorization tree of height* $poly(n)$.

Thanks!